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MATHEMATICS teacher

AUGUST 2012

the bird tetrahedron

// classroom assessment

// geometry for the visually impaired

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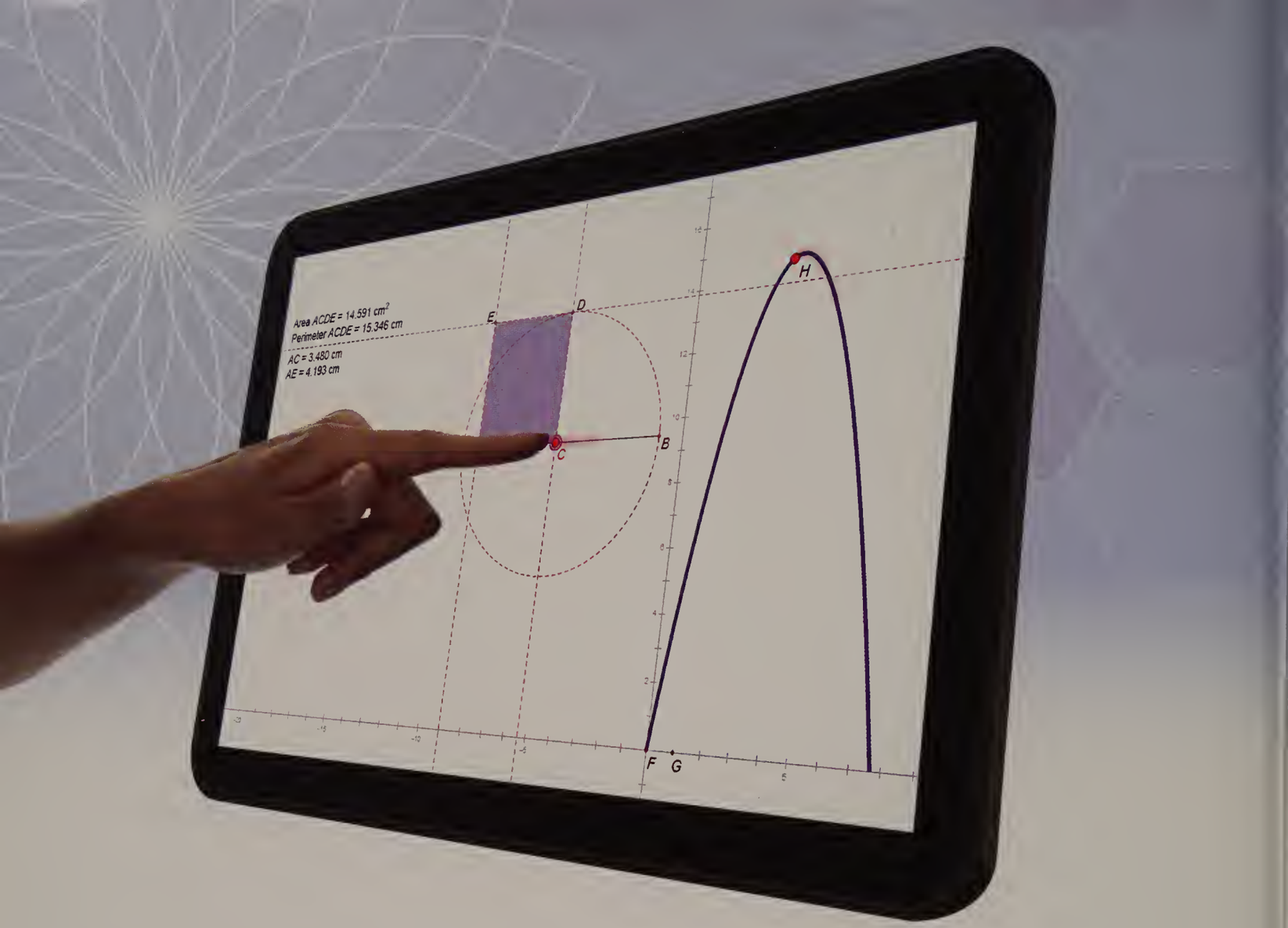

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on the cover

Take flight as *MT* brings you an international perspective on teacher education with Øistein Gjøvik's exposition of how to use an origami construction to explore basic geometry including symmetry, algebra, combinatorics, and problem solving ("Flying High with the Bird Tetrahedron," pp. 16–21). Although Norwegian in origin, this hands-on activity fits well in any high school or teacher education classroom. Come fly with us!

COVER PHOTO: CHUNG LEE/TMG

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MATHEMATICS teacher

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Mathematics Teacher, an official NCTM journal, is devoted to improving mathematics instruction for grades 8–14 and supporting teacher education programs. It provides a forum for sharing activities and pedagogical strategies, deepening understanding of mathematical ideas, and linking mathematics education research to practice. *Mathematics Teacher* solicits submissions from high school mathematics teachers, university mathematicians, and mathematics educators and strongly encourages manuscripts in which ideas relate to classroom practice. Manuscripts previously published in other journals or books are not acceptable. NCTM publications present a variety of viewpoints. The views expressed or implied in this journal, unless otherwise noted, should not be interpreted as official NCTM positions. The appearance of advertising in NCTM's publications and on its websites in no way implies endorsement of approval by NCTM of any advertising claims or of the advertiser, its product, or its services. NCTM disclaims any liability whatsoever in connection with advertising appearing in NCTM's publications and on its websites.

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- "Using Disks as Models for Proofs of Series" (p. 46): Two activity sheets
- Media Clips (p. 8): One additional clip with questions and solutions
- Mathematical Lens (p. 12): More photographs and sketches
- Activities for Students (p. 52): Customizable activity sheets

Download one of the free apps for your smartphone. Then scan this tag to access **www.nctm.org/mt.**

in the next issue

Coming in September 2012:

- "Three-Dimensional Printing: A Journey in Visualization," by Adam Poetzel, Joseph Muskin, Anne Munroe, and Craig Russell
- "Core Conversations with Educative Dragging," by Jeffrey Wanko, Michael Todd Edwards, and Steve Phelps
- "Fielding an After-School Mathematics Lab," by Christina Punches-Guntsch and Erin Kenney

New Times, New Challenges

In *The World Is Flat* (2005), Thomas Friedman sounded an alarm across American culture with his concise portrayal of a rapidly changing world that will directly challenge many traditional practices of American life—in everything from business to education. In many ways, Friedman's twenty-first-century siren is an extension of Albert Einstein's postulate of more than a half century earlier: "We can't solve problems by using the same kind of thinking we used when we created them." Both Friedman and Einstein insist that new times and new challenges require new ideas, relentless analysis, and constant readjustment; in this way, we can ensure that any new process delivers what we need it to.

The *Mathematics Teacher* Editorial Panel agrees that educators must continuously consider the question of what it means to effectively teach the workers and leaders of tomorrow the mathematical facts and skills that they will need to solve problems both present and yet unknown. With every issue of *MT*, the Editorial Panel aspires to present feature articles and departments that spur your thinking, inspire you, and cause you to reflect on your teaching. The goal is to reach students through lessons and challenges that equip them to succeed in our rapidly changing, endlessly challenging world.

The Editorial Panel of *Mathematics Teacher:* (back row) David Custer, Sandra Madden, Greg Stephens, Greg Foley, Stan Izen, Dane Camp; (front row) Laurie Rubel, Dan Canada, Albert Goetz, Allison Golden, Sarah Schuhl.

Mathematics teachers need a forum to discuss the pressing issues that they face every day, and *MT*'s Reader Reflections and On the Front Burner provide an opportunity for collegial dialogue beyond the school's walls. Through Sound Off!, *MT* welcomes salient opinions on educational issues that affect teachers and students. Problem Solvers presents interesting challenges and invites classroom teachers to submit students' solutions for other teachers and students to read and learn from. These departments—along with Media Clips, Mathematical Lens, Activities for Students, Connecting Research to Teaching, Technology Tips, Delving Deeper, The Back Page, and the Calendar—all embody *MT*'s intention to support mathematics teaching and learning now and in the future, with its ever-expanding, ever-uncertain possibilities.

In volume year 106, *MT* readers can also look forward to a special preelection issue in October; the annual Focus Issue in November, with its theme Flexible Mathematical Thinking; and more articles on working with the Common Core State Standards.

Einstein's pronouncement encapsulates the vital necessity of ceaseless invention, whereas Friedman's essays demand that we recognize the blinding speed at which our technological world changes and the human proximity it creates. *Mathematics Teacher* intends to respond to both perspectives by providing relevant, challenging ideas that support all teachers in their effort to improve mathematical instruction for all students. We look forward to hearing from you and sharing your thoughts with *MT*'s readers.

REFERENCE

Friedman, Thomas L. 2005. *The World Is Flat: A Brief History of the Twenty-first Century.* New York: Farrar, Straus and Giroux.

THE DISCUSSION continues

Excerpts from Sid Soni's original letter (the complete text of which appeared in *MT* March 2012), several responses to the original letter, and Soni's reply to those responses follow.

THE CASE FOR REPEATING ALGEBRA 1

At midyear in my algebra 2 course, I distributed my students' grades, which ranged from the 40s to the 90s. I felt that the failing students had simply not mastered the prerequisite algebra skills required to perform in this course, ... so I administered an informal diagnostic to see just how closely my students' first-year algebra skills correlated with their second-year algebra performance....

The correlation between the students' scores on the algebra skills diagnostic and their algebra 2 midyear average was $r = .75$. Further, the bottom 20th percentile (those students scoring 44% or less) correlated to a 100% fail rate in the algebra 2 course at midyear. This was strong evidence that this subset of the class simply did not have the prerequisite algebra skills to pass algebra 2. What steps can be taken to avoid this situation in the future?

... Courses with axiomatic prerequisite skills should require students to pass a basic entrance exam....

The ostensible choice is either to promote the student into a class he or she probably cannot pass or to give him or her another chance to master the basics under different circumstances—having a different teacher, being one year older, seeing the material again, and so on. For students who cannot demonstrate algebra skills, I believe that repeating first-year algebra is the best recourse....

One rationale for promoting borderline students is to ensure that they complete four years of high school mathematics, thus bolstering their college applications....

Weak students, despite having taken four years of mathematics, will not likely qualify for the caliber of college that encourages doing so.

Further, at many colleges, particularly less competitive ones, incoming students are given a placement exam that is heavily skewed toward measuring their algebraic skills....

If I had to wager on the outcome of this exam, I would predict that this cohort will be retaking algebra in college, thus begging the question: Are these students better off repeating algebra in grade 10 or in grade 13?

Sid Soni
siddsoni@hotmail.com
Somers High School
Westchester County, NY, Feb. 24, 2011

REACTIONS FROM READERS

I agree completely. I teach algebra 2 and have seen exactly what you describe, although many of my students are even less prepared. Not only have they no concept of algebra 1 skills, especially factoring, but also many of my juniors and seniors are completely lacking in basic arithmetic facts and have no number sense at all. They need to return to fourth grade and repeat all mathematics from there.

Thank you for a great letter. I am going to share it with my colleagues.

Glen Cooke
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Cookeville High School
Cookeville, TN, Mar. 2, 2012

We appreciate the interest and value the views of those who write. Readers commenting on articles are encouraged to send copies of their correspondence to the authors. For publication: All letters for publication are acknowledged, but because of the large number submitted, we do not send letters of acceptance or rejection. Letters to be considered for publication should be in MS Word document format and sent to **mt@nctm.org**. Letters should not exceed 250 words and are subject to abridgment. At the end of the letter include your name and affiliation, if any, including e-mail address, per the style of the section.

I agree with the gist of the letter. I teach at a vocational high school, and we struggle with the overall low and inconsistent algebra readiness skills of our rising ninth graders. We are currently revamping our four-year mathematics paths to emphasize college readiness for all our students. All entering students will take algebra 1, even if they had it in middle school. Because our students have a wide array of mathematics skills and backgrounds, we will offer two levels of the course—basic and advanced—and will differentiate within each. Ninth graders who are designated as Title I students will also take a separate mathematics lab class that will provide remedial instruction in prerequisite skills.

Given my school's goal of ensuring college readiness, I firmly believe that students should not be able to move on to algebra 2 until they are ready. I would like to try to institutionalize the main idea behind Soni's proposition. This would mean giving students a test toward the end of algebra 1. Students who do not pass this test would have to attend summer school. If they can demonstrate mastery, they would move on to the next algebra course their sophomore year. Otherwise, they would repeat algebra 1.

Diana Keohane
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Greater Lowell Technical High School
Tyngsborough, MA, Mar. 6, 2012

I read with interest the rationale for requiring high school students to pass an entrance exam to be allowed to take algebra 2 and, if they fail, to repeat algebra 1. I teach both courses at a small alternative high school. Most of my students have had little success in mathematics, often because of an inability to complete homework. Many fear the subject. Our school district requires four years of high school mathematics to graduate, and that requirement seems to be more common across the nation. Forcing students to retake algebra 1 would require them to double up on mathematics in their senior year or to attend summer school for credit recovery.

The issue of students not acquiring basic algebra skills the first time around seems to me to be more a teaching issue than something to blame students for. The mathematics textbook publishers have included every early algebra skill possible in algebra 1 textbooks to comply with the myriad state standards that currently exist. Students suffer because many teachers feel compelled to "get through the book," so there is no learning of any skill at a deep level. If students happen to be blessed with natural talent in logical-mathematical intelligence, they can maneuver the flyby of algebra 1 and then succeed in algebra 2 handily. If not, they must struggle through the rest of their high school career with a poor understanding of what I agree is the most important mathematics class for future success. It appears that the Common Core State Standards are addressing this issue by narrowing the curriculum and ensuring that all states are working from the same standards. This approach will allow textbook publishers to tighten their scope and sequence for each high school mathematics course.

We teachers also need to examine our own practice. Are we designing classroom activities that require students to discuss what they need to learn? Are we asking them to synthesize and reflect? Are we ensuring that our curriculum is not a mile wide and an inch deep but instead staying with concepts long enough for deep understanding to occur? I have designed my algebra 1 curriculum as a two-year course to ensure success. Could students be identified early for this type of intervention yet still have time to complete four years of mathematics, including algebra 2?

It seems unfair to have students pay the price for adjustments that need to happen on our end. If students are not ready, isn't it our job to help them get ready?

Susan Mick
smick@tcsd.org
Summit High School
Jackson, WY, Mar. 11, 2012

I do not disagree with Soni's assertion that some students need to spend another year in some mathematics classes. I do disagree with his assumption that students who do not master algebra 1 skills by the end of that course will never learn them.

Most recent mathematics textbooks do not assume that students have mastered the material from the previous course. Surveying the algebra 2 textbooks in my office, I note that many start with a chapter reviewing algebra 1. (An older textbook, Foerster's [1st ed., 1980], starts with "The Field Axioms.") In textbooks for later courses, perhaps half the exercises have a glimmer of the current topic, but these exercises are really there for review. Exercises in verifying trigonometry identities are disguised practice in adding fractions, finding differences of squares, and factoring. Witness the engorgement of Lang's *A First Course in Calculus,* which has ballooned from 250 pages to 700 pages between the first edition (1964) and the fifth (1986), even adding an entire section entitled "Review of Basic Material."

I hope that students leaving my courses have mastered the material, but my realistic expectation is that in their next mathematics course they will vaguely recollect the topics I covered. It is in the next course, when they have to apply those skills, that they will master them. Many academic majors in which students will never use the calculus nonetheless require students to take it, because that's where students master algebra.

I do not expect students coming into my algebra 2 course to have mastered algebra 1. I have learned not to expect that they can add fractions. Many are only vaguely familiar with addition with carrying and subtraction with borrowing. I have witnessed students using long multiplication to multiply by 10. Do I send them back to fourth grade?

Students who do not do well in one course will struggle that much harder in the next. For many students, the best approach is to repeat a course.

It is not true, however, that a student who did not master several

algebra 1 topics while in that course will not be able to improve those skills during the next course. The system is designed to support that student because most students are that student.

J. Bradford Burkman
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Louisiana School for Math, Science, and the Arts
Natchitoches, LA, Mar. 13, 2012

Current practice is to require all students to learn mathematics at the same rate. This practice is unreasonable and results in many students concluding that they can't do mathematics. And it doesn't start with algebra 1. I'm in favor of an approach offered by the Khan Academy, where students, at home, view and re-view a video that presents a skill. The students then do "homework" in class, where the teacher and other higher-achieving students help the slower students individually.

Algebra 1 should not be a 25-student, 180-day event; algebra 1 should be the time it takes for a student to achieve at a level that ensures success at the next level. If we believe that all students can learn some particular subset of mathematics that is deemed important, then we need to give them some control over the time that they individually require for its acquisition. And we need to help them become aware of how they best learn.

Adults acquire new knowledge and skills by teaching themselves. When challenged to learn something new, mature learners are armed with a sense of how they best learn. They make a choice about whether to read, listen, or view with available technology. They confirm their newly acquired learning through discussions with others and by helping others. Many of us have recently learned to use a smartphone, iPad®, and SMART Board™. Each of us acquired these new skills by using learning strategies of our choice, with complete control over our rate of learning.

Jack McCabe
jmccabe@chury.org
Canterbury School
New Milford, CT, Mar. 19, 2012

Editor's note: This year, McCabe celebrates his fiftieth year of membership in NCTM.

I read the letter in *Mathematics Teacher* about second-year algebra students who would be better off repeating first-year algebra. Please send me a copy of the basic entrance exam that you gave your algebra 2 students.

I would like to share this letter and your exam with my department.

Chris Lodes
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Lincoln Northeast High School
Lincoln, NE, Mar. 3, 2012

SID SONI RESPONDS

I did not want my letter to be just another unconstructive "The kids can't factor!" mathematics teacher cliché. A gratifying postscript was receiving the varied responses from other high school teachers who felt that I provided the impetus for taking action at the local level. Others might feel that repeating the entire course presents a false dilemma, as Burkman's response indicates. A different plan of action would be to identify an explicit subset of core exit skills for algebra students. These "must have" skills constantly arise in later courses (e.g., solving linear equations, proportional reasoning). During the year, these skills can be emphasized, reinforced independently, drilled and killed, reassigned as summer work, or retested before promotion to the next level. These additional opportunities would make students' chances of mastering the content and succeeding in second-year algebra that much higher.

Should there be shame in repeating a course if it didn't click the first time? I say no. Complex learning can take hundreds of cycles of repetition to nail down. Mathematics, dance steps, golf swings, musical scales, wine tasting, photography—none of these is a "teach-once, learn-once" skill. In any nontrivial learning process, we need to step back, practice, reprocess, revisit, synthesize, and sometimes drill and kill. Only when we "own" a stage will we get anything out of the following stage.

The larger implications tied to high school algebra ability are worth noting. Failing to master this coursework can affect all future mathematics and science coursework and will even limit the types of college majors (and, hence, careers) that students may pursue. High school algebra skills include, among others, interpreting symbolic notation; manipulating expressions, formulas, and variables; solving equations; understanding probability; and graphing in the Cartesian plane. These skills are necessary for success in college courses as well. Statistics, calculus, and economics, which make use of data, formulas, equations, and graphing, require these skills. Academic areas that have an explicit statistics or calculus requirement include psychology, nursing, accounting, economics, social work, business management, marketing, finance, chemistry, biology, computer science, physics, and engineering. Clearly, not mastering algebra skills can close numerous academic and professional doors.

One pace does not fit all, which is why McCabe's response is so vital. My letter was written to address a core limitation of the traditional classroom model (i.e., limits of individual pacing). A bigger picture emerges as the very nature of education and autodidactic learning evolves. How will self-paced online learning coalesce or clash with the existing model of education? Instead of a literal interpretation of "repeat the course," the universal value of this discussion is to underscore the importance of adequate foundational prerequisite skills in any sequential learning process. This principle holds true whether the learning occurs in a classroom, on a computer, or during personal growth and scholarship.

MEDIA clips

Ron Lancaster

13th Birthday

Happy $1 + 3 + 3^2$th Birthday, Andrea!

Triskaidekaphobia! No, we have no fear.
Andrea, our new teenager, is here.

We love you and wish you all the best.

Dad, Mom, Greg, Jessica

Source: Birthday announcement, *Hamilton (Ontario, Canada) Spectator*, October 8, 2011, http://www.lifenews.ca/thespec/profile/223345

1. (*a*) Working with only 1s and 4s, create several expressions that could be used in an announcement for Andrea's 14th birthday.

   (*b*) Find the next two times when Andrea's age will be the sum of a finite geometric series. Each series will have at least three terms and will begin with 1.

   (*c*) How old will Andrea be when her age can be written in two different ways as a sum of a finite geometric series? Both series will have the first term equal to 1.

2. (*a*) We know that October 8, 2011, was a Saturday. Find the day of the week for the dates in **table 1 ("13th Birthday")**. Using the facts that a non-leap year has 365 days (52 weeks and 1 day) and that a leap year has 366 days, develop a pattern for how the day of the week changes from one year to the next. (Do not rely on the Internet when answering this question.)

   (*b*) Find the next two times after the year 2011 when October 8 occurs on a Friday.

Table 1 ("13th Birthday")

Date	Day of the Week
October 8, 2012	
October 8, 2013	
October 8, 2014	
October 8, 2015	
October 8, 2016	
October 8, 2017	
October 8, 2018	
October 8, 2019	
October 8, 2020	
October 8, 2021	

Media Clips appears in every issue of *Mathematics Teacher*, offering readers contemporary, authentic applications of quantitative reasoning based on print or electronic media. All submissions should be sent to the editors. For information on the department and guidelines for submitting a clip, visit http://www.nctm.org/publications/content.aspx?id=10440#media.

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No Triskaidekaphobia Here!

People watchers, bikers and motorcycle enthusiasts descended on Port Dover today for the 50th Friday the 13th festival.

Elbow room was in short supply as a litany of sun-seekers came to look—and be seen.

Among them were an estimated 20,000 bikes sprinkled throughout the sun-drenched streets.

Although some took to the beach to escape the soaring temperatures, most were happy to stay in the crowded street where the action was.

Organizers, who estimated more than 3,000 garbage bags would be thrown away today, expected it to be the biggest gathering in the event's history, topping 150,000 people.

This comes after the founding event took place with a party of 25 bikers.

Said Bill Dainard, who has made it out to every Friday the 13 for the past six years: "The camaraderie is the best part. There's no pressure, there's no schedule. We just walk around, look at bikes, look at people and meet friends."

Source: Jeremy Grimaldi, "Thousands of Bikers Converge on Dover," *Hamilton (Ontario, Canada) Spectator*, Friday, August 13, 2010

1. (*a*) Find the number of Friday the 13ths for each of the years in **table 1** (**"No Triskaidekaphobia Here!"**).

   (*b*) Suppose that the journalist who wrote this article will be asked to write a similar article to be published on Friday, November 13, 2015. Assuming that the event in Port Dover does take place, what part or parts of the article could the journalist write today?

   (*c*) Visit the website www.pd13.com and determine in what year the 100th anniversary of the Friday the 13th event will take place. Before doing so, estimate what year this will be.

Table 1
("No Triskaidekaphobia Here!")

Date	Number of Friday the 13ths
2011	
2012	
2013	
2014	
2015	

Media Clips

Read any good mathematics lately?

If you have read, seen, or heard some intriguing mathematics in the media recently, perhaps the item led you to extrapolate some questions about the mathematics. Or perhaps the mathematics itself made you wonder, "Can these numbers really be right?"

Compose some questions about the mathematics presented in the item and send them to the editors of Media Clips. Or just submit the item, and the editors will create the questions.

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MEDIA CLIPS solutions

"13th Birthday" answers

1. (*a*) Two examples are $1 + 1 + 4 + 4 + 4$ and $\sqrt{4} \cdot (1 + 4) + 4$.

   (*b*) The next two times when Andrea's age will be the sum of a finite geometric series with at least three terms and the first term 1 will be when she is 15 years old $(1 + 2 + 2^2 + 2^3)$ and when she is 21 years old $(1 + 4 + 4^2)$.

   (*c*) Andrea will be 31 when her age can be written in two different ways as a sum of a finite geometric series with the first term 1. The two representations are $(1 + 2 + 2^2 + 2^3 + 2^4)$ and $(1 + 5 + 5^2)$.

2. (*a*) Because 2012 is a leap year, October 8, 2012, will occur 366 days (52 weeks and 2 days) after October 8, 2011. We can ignore the 52 weeks because they do not have any effect on the day of the week (if today is a Friday, a week from today or 1000 weeks from today will be a Friday). October 8, 2012, will occur on the day of the week that is 2 days after the day of the week on which October 8, 2011, occurred. That was a Saturday, so October 8, 2012, will be a Monday.

   Because 2013 is not a leap year, October 8, 2013, will occur 365 days (52 weeks and 1 day) after October 8, 2012. Once again, we can ignore the 52 weeks. October 8, 2013, will occur 1 day after the day of the week on which October 8, 2012, will occur. That day is a Monday, so October 8, 2013, will be a Tuesday.

   The day of the week will continue to advance by 1 day until the next leap year, when it will move ahead by 2 days. This pattern will continue indefinitely.

Table 2 ("13th Birthday")

Date	Day of the Week
October 8, 2012	Monday
October 8, 2013	Tuesday
October 8, 2014	Wednesday
October 8, 2015	Thursday
October 8, 2016	Saturday
October 8, 2017	Sunday
October 8, 2018	Monday
October 8, 2019	Tuesday
October 8, 2020	Thursday
October 8, 2021	Friday

   (*b*) From **table 2 ("13th Birthday")**, we see that the next year after 2011 in which October 8 will be a Friday will be 2021. If we use the pattern discussed in part (*a*), the next year after 2021 in which October 8 is a Friday the 13th will be 2027.

"No Triskaidekaphobia Here!" answers

1. (*a*) The number of Friday the 13ths can be seen in **table 2 ("No Triskaidekaphobia Here!")**.

   (*b*) The event took place 10 times (see **table 2**) after the 50th anniversary on August 13, 2010; thus, the journalist would know for certain that November 13, 2015, will be the 60th anniversary of the event. If the journalist had access to data from previous years in which November 13 was a Friday, then it would be possible to estimate the size of the crowd and predict the weather for November 13, 2015. But it would not be possible to state the size of the crowd or the weather for that day until the event had taken place.

Table 2 ("No Triskaidekaphobia Here!")

Date	Number of Friday the 13ths
2011	1
2012	3
2013	2
2014	1
2015	3

   (*c*) The 100th anniversary of the event will take place on Friday, January 13, 2040.

The following website was used to verify answers for the questions posed by this clip: http://www.calendar-365.com/2012-calendar.html.

RON LANCASTER, ron2718@nas.net, teaches mathematics education courses at the Ontario Institute for Studies in Education at the University of Toronto, Ontario.

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For an additional clip with questions and solutions, download one of the free apps for your smartphone and then scan this tag to access **www.nctm.org/mt022.**

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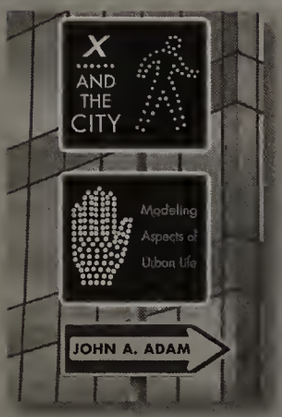

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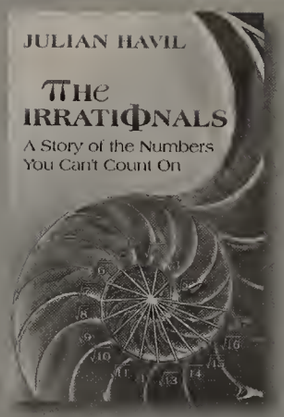

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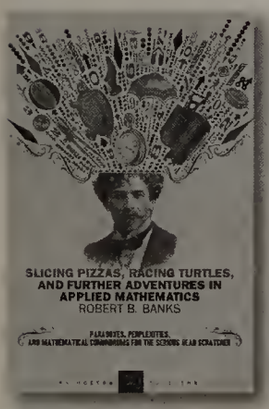

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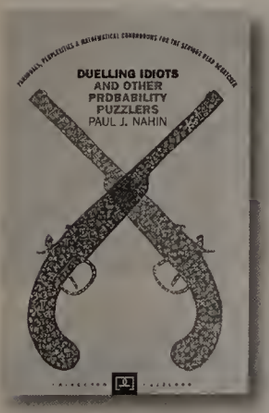

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Thinking outside the Cube

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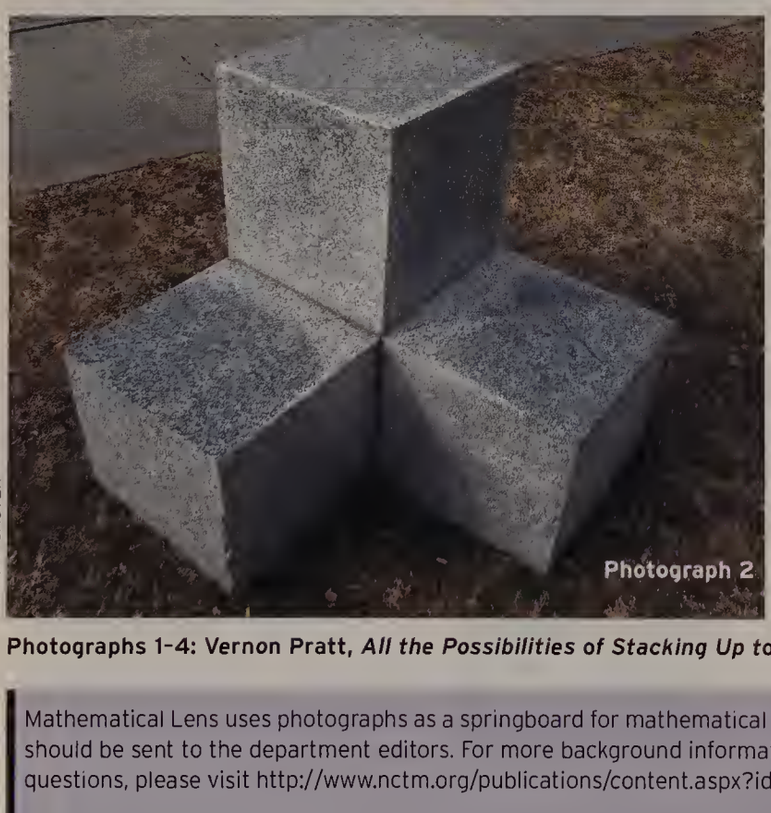

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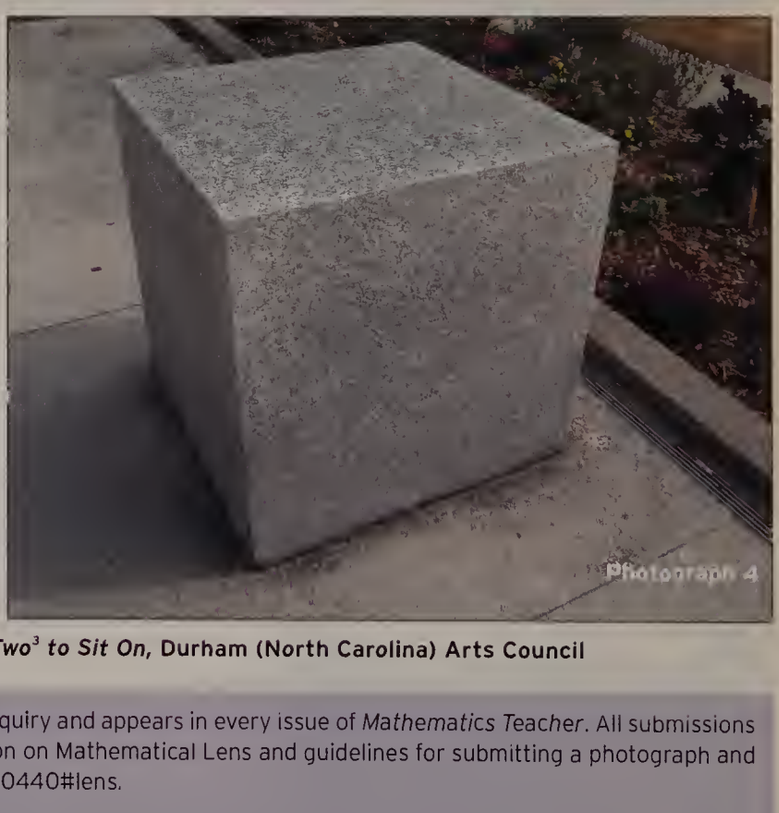

RON LANCASTER

Photographs 1–4: Vernon Pratt, *All the Possibilities of Stacking Up to Two^3 to Sit On*, Durham (North Carolina) Arts Council

Mathematical Lens uses photographs as a springboard for mathematical inquiry and appears in every issue of *Mathematics Teacher*. All submissions should be sent to the department editors. For more background information on Mathematical Lens and guidelines for submitting a photograph and questions, please visit http://www.nctm.org/publications/content.aspx?id=10440#lens.

Edited by ***Ron Lancaster,*** *ron2718@nas.net, University of Toronto, ON, Canada, and* ***Brigitte Bentele,*** *brigitte_bentele@trinityschoolnyc.org, Trinity School, New York, NY*

Photograph 5
Impermissible stacking

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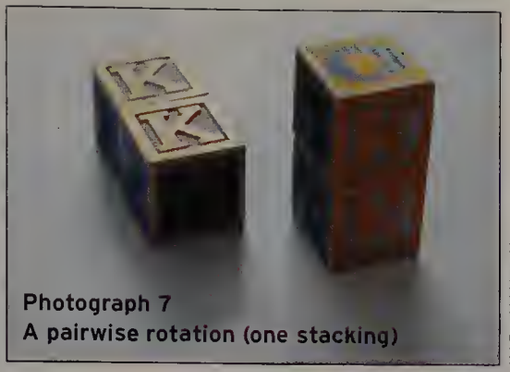

Photograph 7
A pairwise rotation (one stacking)

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Photographs 6
Impermissible stacking

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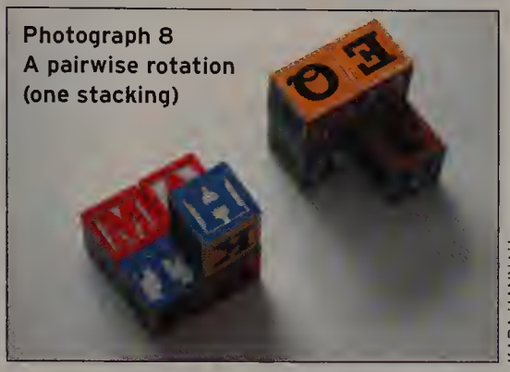

Photograph 8
A pairwise rotation
(one stacking)

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Photograph 9
The Cube,
Indianapolis

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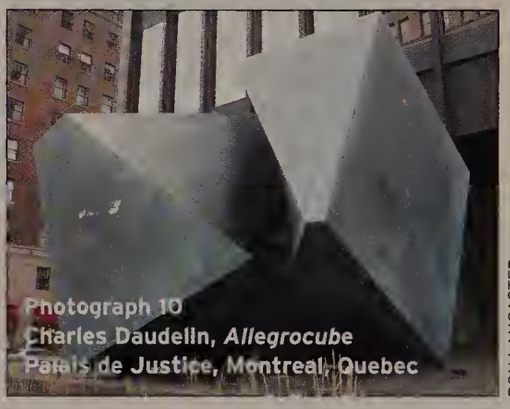


Photograph 10
Charles Daudelin, *Allegrocube*
Palais de Justice, Montreal, Quebec

RON LANCASTER

The Durham (North Carolina) Arts Council displays a sculpture group (see **photographs 1–4**) by Vernon Pratt, a former professor of art at Duke University, on its front lawn. The work's suspected mathematical connection is made explicit in its title: *All the Possibilities of Stacking Up to Two³ to Sit On* (1988).

Pratt created each sculpture from identical granite cubes arranged to fit within a $2 \times 2 \times 2$ space and aligned so that any cubes in the sculpture met either face to face or edge to edge; unsupported cubes were not allowed. Pratt also considered sculptures that are rotations or reflections of each other as the same and excluded duplicates. Some examples of arrangements that Pratt did not allow are shown in **photographs 5** and **6**; **photographs 7** and **8** show duplicates.

1. (*a*) Determine how many sculptures Pratt created for the collection. Find this total in as many different ways as you can. Consider drawing these sculptures or creating them using wooden blocks or other manipulative or technology. First, estimate how many cubes will be needed.

   (*b*) Count the number of cubes used to make all the sculptures and compare this result with your estimate from part (*a*).

2. Kara Hannah, a high school student interested in photography, devised the following question after using a set of wooden blocks to make the entire collection: Suppose that Pratt had decided that each sculpture had to fit within a $2 \times 2 \times 3$ space. Find the total number of sculptures that would be in this collection. Do not include rotations of sculptures in your total count.

3. (*a*) Without using any technology, determine whether the number of cubic feet at The Cube in Indianapolis (see **photograph 9**) is a cube of an integer (commonly called a perfect cube).

   (*b*) In his book *Meditationes Algebraicae* (1770), Edward Waring wrote, without proof, that every positive integer can be written as a sum of nine or fewer positive cubes. For example, $9 = 1^3 + 2^3$ and $23 = 2^3 + 2^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3$. Without using any technology, express the number from part (*a*) as a sum of nine or fewer positive cubes.

4. The sculpture in **photograph 10** was created by Charles Daudelin, a Canadian sculptor and painter and a major Québécoise artist. The sculpture, entitled *Allegrocube* (1973), is located in front of the Palais de Justice in Montreal. The cube has sides of 2.4 m in length and is made of "Muntz metal," an alloy of copper and zinc. The two halves of the structure can be moved by means of an electric motor to form a closed cube. Whether the structure is open or closed is directly connected to its location. What is this connection?

MATHEMATICAL LENS solutions

1. (*a*) Hannah correctly found that the collection comprises fifteen different sculptures (see **photograph 11**). Hannah explained that her methodology was to start with the large $2 \times 2 \times 2$ sculpture and keep removing cubes one by one to form new arrangements, being careful not to miss one. (To see Hannah's methodology, go to **www.nctm.org/mt023**.)

Brigitte Bentele took a different approach to building the collection. She first considered all the possible arrangements with 4 cubes on the base. Then she considered those that have 3 on the base that are not rotations of ones that have already been constructed and so on (see **fig. 1**).

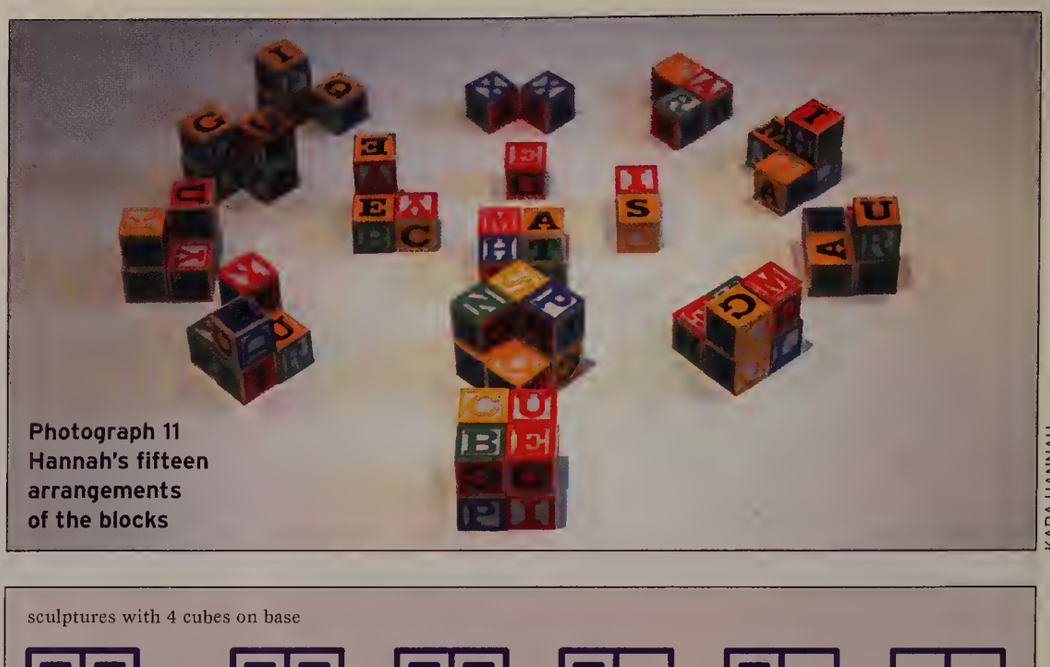
Hannah's and Brigitte's work notwithstanding, a Google Earth™ view of the collection shows sixteen sculptures. The Durham Arts Council reports that there are only fifteen sculptures; however, a plaque lying on the ground in the middle of the collection accounts for sixteen objects being visible from above. Many years ago, Albert Watson, then a mathematics teacher at the Fieldston School in New York City, created a collection of drawings of the sculptures; he felt that the plaque should be considered part of the collection because it plays the role of the null set. If we include the plaque, then the collection consists of sixteen sculptures. Watson's sketches include all the impossible sculptures along with a naming scheme for the entire collection. (To see Watson's sketches, go to **www.nctm.org/mt023**.)

It is difficult to estimate how many cubes will be needed before making the models.

(*b*) Pratt used 64 cubes to make the collection of granite sculptures.

Pratt died in 2000 at the age of fifty-nine after a mysterious bicycle accident in Virginia. Details and a tribute can be found at these websites:

- http://dukechronicle.com/article/community-mourns-loss-thoughtful-artist
- http://dukechronicle.com/article/art-professor-dies-after-dec-accident



Photograph 11
Hannah's fifteen arrangements of the blocks

KARA HANNAH

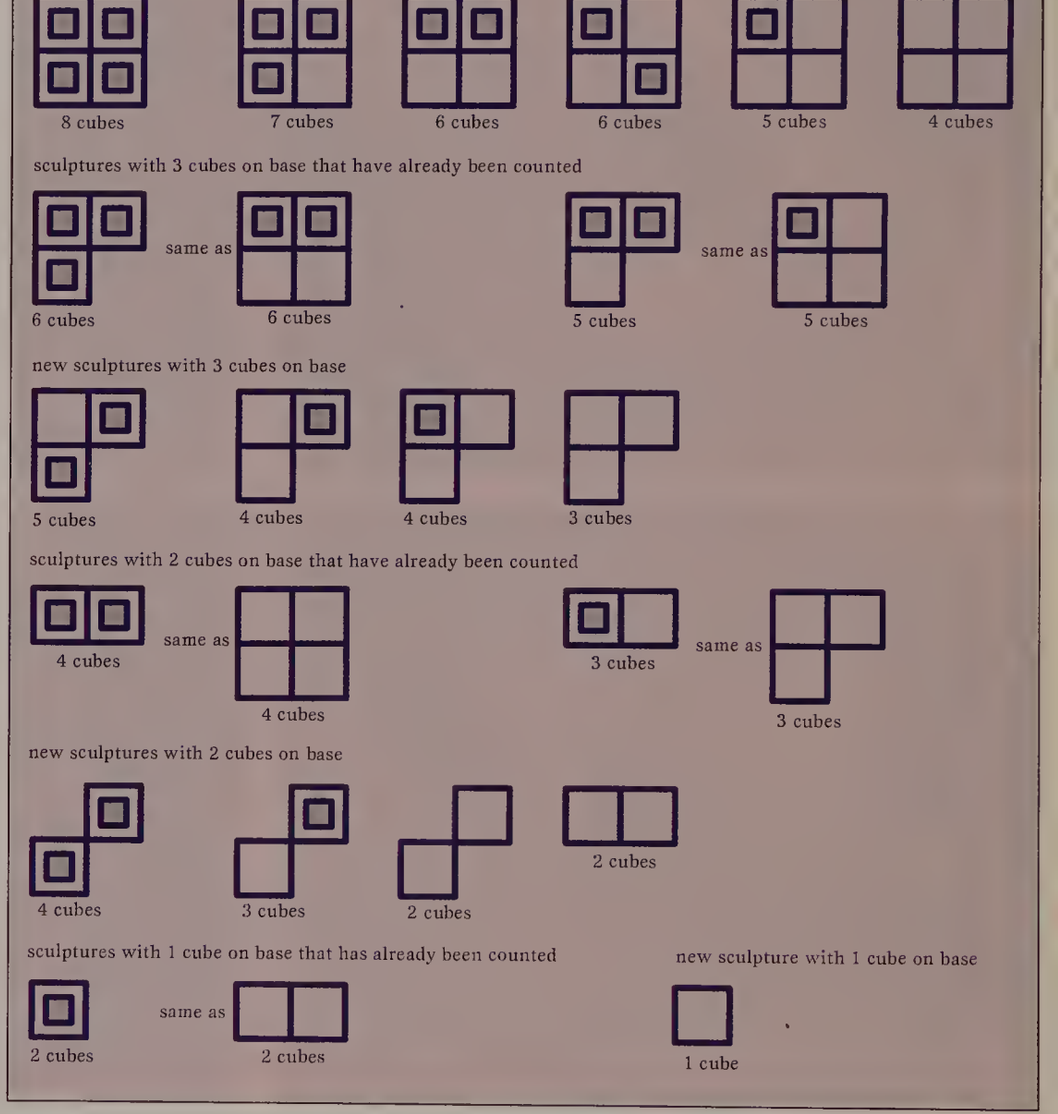


Fig. 1 Starting with eight cubes and removing them is another way to approach the problem.

2. Readers are encouraged to use Bentele's strategy (or one of their own) to find the total number of sculptures and to submit their totals to the Mathematical Lens editors for possible publication in *MT*. Start with

a base of 6 cubes (there is only one case) and find the number of sculptures that can be made by adding 1 cube, then 2 cubes, and so on. Then work with the sculptures that have a base made up of 5 cubes (there are two cases), then a base of 4 cubes (there are five cases), and so on.

Readers may also want to consider extending the problem of counting the number of sculptures for spaces such as $2 \times 2 \times 4$, $2 \times 2 \times 5$, and so on.

3. (*a*) Note that 360,000 is equal to $360 \cdot 10^3$. Because 360 is not a perfect cube (it is between 7^3 and 8^3), we know that 360,000 is not a perfect cube. Too bad—otherwise, The Cube could have been located in a cube!

   (*b*) Instead of working with the number 360,000, let's see what we can do with the number 360. Because 7^3 is equal to 343, that leaves us with 17 to work with. But 17 can be written as $1^3 + 2^3 + 2^3$. So $360 = 1^3 + 2^3 + 2^3 + 7^3$; therefore, $360{,}000 = 10^3 + 20^3 + 20^3 + 70^3$.

Information about Waring's experiments and problem can be found at the following websites:

- http://en.wikipedia.org/wiki/Cube_%28algebra%29
- http://www.maa.org/mathland/mathtrek_07_19_04.html
- http://en.wikipedia.org/wiki/Waring%27s_problem

4. Readers may have struggled to figure out how the state of the cube (open or closed) was connected to its location. The key is to think outside the box (see **photograph 12**) and focus on what takes place in the Palais de Justice. The artist has used a geometric shape to demonstrate the difference between freedom (depicted by open halves) and confinement (depicted by closed halves). The halves of the cube are opened when the court issues a verdict of not guilty and are closed when the court issues a verdict of guilty.

   When opened, the sculpture is a contrast of straight lines and curved surfaces. Its size and weight are a silent testament to the power of justice, and its two forms refer to the legal system's ability to confine and set free.

KARA HANNAH

Photograph 12 The key to the solution

Additional information about Charles Daudelin and his sculpture *Allegrocube* can be found at these websites:

- http://wikipedia.qwika.com/en2fr/Allegrocube
- http://www.metrodemontreal.com/art/daudelin/index.html
- http://www.inuitfinearts.com/vincentgallery/bronzes/daudelin/DaudelinC-Cube.html
- http://www.thecanadianencyclopedia.com/articles/charles-daudelin

KARA HANNAH is a student at Bishop Tonnos Secondary School in Ancaster, Ontario. She is interested in design, drawing, and photography.

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For Hannah's block arrangements and Watson's sketches plus a photograph for your own mathematical questions, download one of the free apps for your smartphone and then scan this tag to access **www.nctm.org/mt023**.

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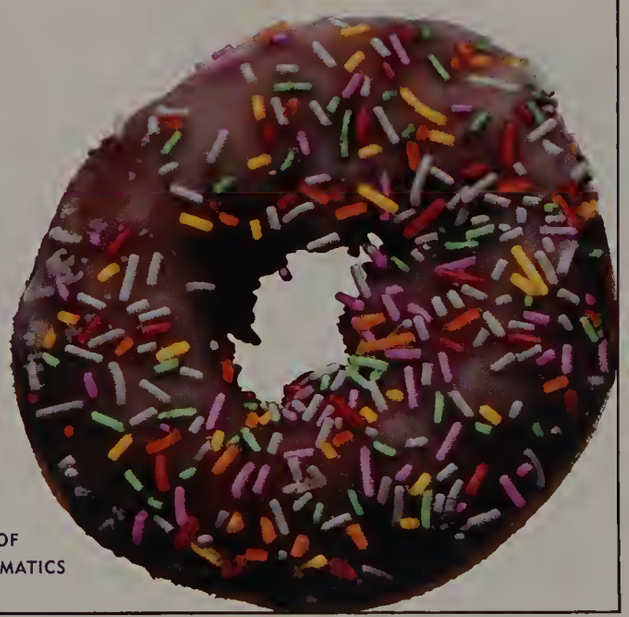

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FLYING HIGH WITH THE

TETRAHEDRON

An origami activity can lead to rich tasks in several branches of mathematics.

Øistein Gjøvik

A good problem, I once read, is one that keeps on giving and giving. One of my favorite activities holds within it problems that provide different perspectives and rich mathematical challenges time and time again.

This activity is a good first step for preservice teachers to get to know one another, start talking to one another, and start discussing mathematics. It can also show them how mathematics teaching and learning have changed to a more practical, explorative, and collaborative approach, which may be different from what they encountered as students in Norwegian schools. I will describe how I use this activity with my students and share some of their questions and contributions.

Many of the tasks and activities presented here can readily be adapted to secondary school classrooms, both in Norway and in the United States. My goal is not to provide activities that teachers can copy and use in their own teaching but rather to develop their skills in thinking mathematically, exploring patterns, and being aware of possibilities for developing activities of their own.

FOLDING THE BASIC SHAPE

Begin with sheets of paper in four different colors and hand out one sheet of paper to each student. In Norway, A4 is the standard paper format, but 8 1/2-by-11-inch paper will work well. Each student will make one basic shape from his or her sheet. The final shape, as shown in **figure 1**, will be assembled from three such shapes.

I show students the folding procedure for the basic shape (see **fig. 2**). Readers might find it helpful to make the shape before continuing because it has properties difficult to detect from a photograph. An interactive look at the shape can be found using the free software Poly (http://www.peda.com/polypro/) or checking the Mathworld website (http://mathworld.wolfram.com/TriangularDipyramid.html).

As we work, we talk about which mathematical concepts we encounter as we make the folds.

ØISTEIN GJØVIK

Fig. 1 The finished bird tetrahedron is composed of three smaller shapes.

We first make a square from an A4 sheet, a process familiar to most students. To get students to begin thinking together, I ask questions such as, "How can we be sure that this is a square?" I also ask them to name as many familiar mathematical concepts as they can find as they fold. In response, students mention quadrilaterals, diagonals, triangle height, bisectors, right angles, and isosceles triangles, among others. In an aside, I show students how folding the same square paper in half diagonally leads to an important idea about how to explain and understand the standard formula for the area of a triangle. And in another aside, students learn that folding the three corners of a triangle in toward the base of the height gives an idea for understanding why the angular sum of a triangle is 180 degrees, a result that they will explore later in their studies.

ASSEMBLING THE PIECES

After each student has made one piece of the tetrahedron, each one finds two other students, preferably with sheets of different colors, and together they assemble the three basic shapes into

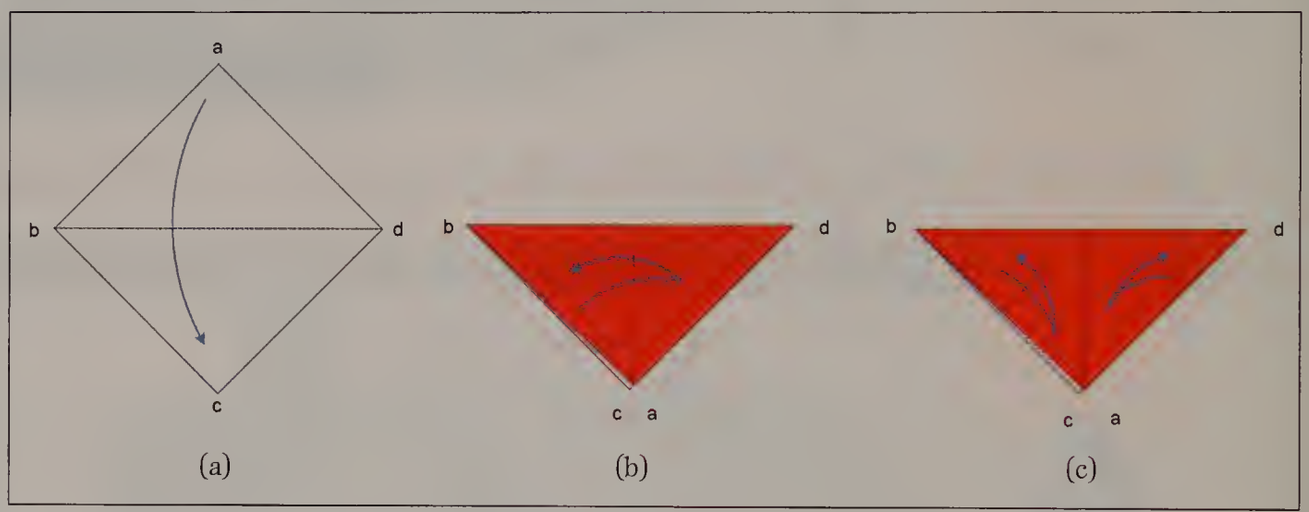


Fig. 2 The basic shape is obtained by making the indicated folds.

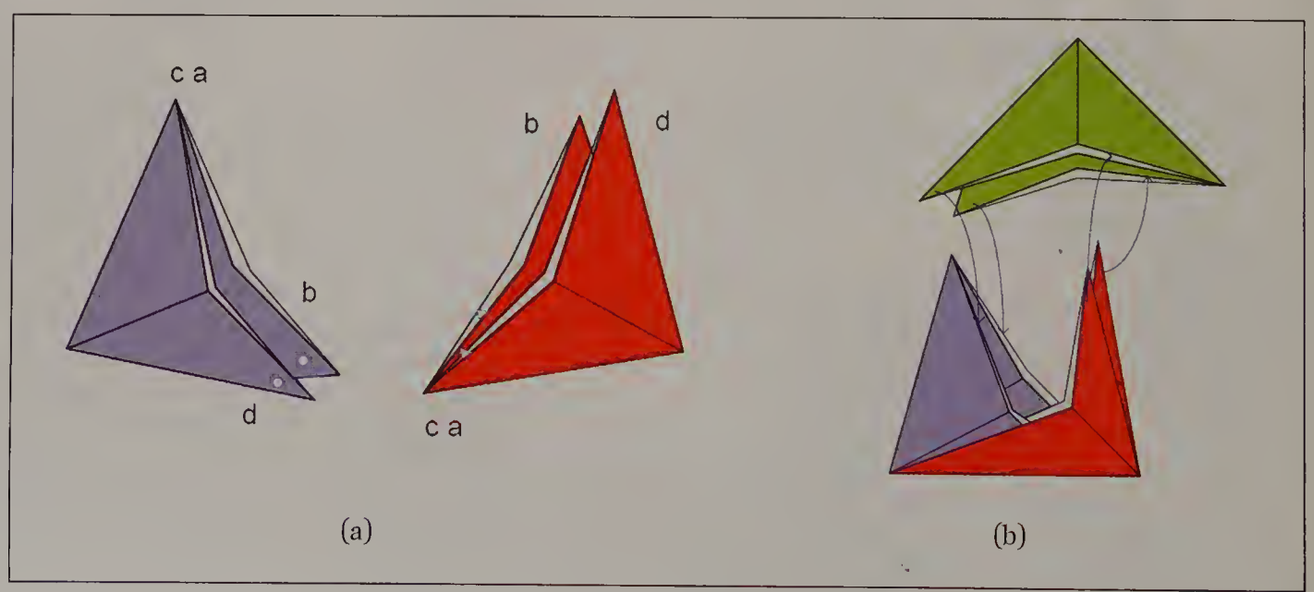


Fig. 3 Assembling the three pieces usually does not present a challenge to students.

the finished shape (see **fig. 3**). Assembling the pieces is easy enough for most students to manage, although some struggle with getting the last piece in place.

PROPERTIES OF THE SHAPE

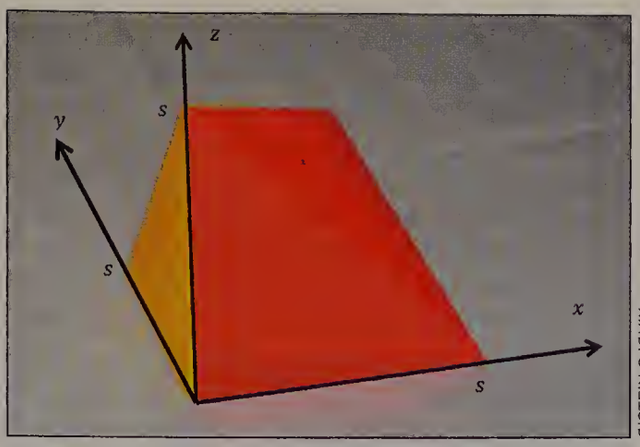
Now, with the finished three-dimensional model in front of us, we again have a whole-class discussion. Questions such as "How many types of triangles can we find?" and "How many different sizes of angles can you see?" help us focus on properties, not just mathematical terms. A typical class has around seventy students, so usually many concepts are mentioned. We make a list on the board of all the mathematical concepts and names that we have observed when examining the shape. Students can contribute even if they are a bit unsure of the mathematics involved. One question that often comes up is whether a triangle with angles of 45, 45, and 90 degrees is an isosceles triangle or a right triangle. Pondering these questions helps students return to the language of mathematics. For students most familiar with teacher-centered mathematics lessons, these activities can serve as a stepping stone toward the discovery, or exploration, culture in today's mathematics classrooms.

Students are now asked to name the shape that they know as a hexahedron. The shape can be seen as made of two pyramids with a common regular, triangular base. Therefore, both *triangular dipyramid* and *double triangular pyramid* are suitable names. However, I explain that the name *bird tetrahedron* is also used. I hold the shape up and ask why. One suggestion is that when the common triangular base is horizontal, the figure is reminiscent of a beak. And it looks quite like a pyramid or a tetrahedron (without being one). Perhaps this is why the strange and "wrong" name has stuck.

CALCULATING THE VOLUME

Students now calculate the volume of the shape. They do so by measuring the sides and calculating the volume of the two pyramids first by using the standard formula. A natural follow-up question is for students to find the algebraic formula for the volume.

Finding the algebraic formula raises the question about variables. Is there a natural "height" or "base"? We start by finding the area of the common plane between the two pyramids, then the height of one of the two pyramids, and finally the volume of this pyramid. The volume of the entire bird tetrahedron will be twice the volume of one pyramid. The calculation is rather lengthy but not too difficult (see Gjøvik [2005], where these ideas were originally explored). The calculation involves using properties of triangles with 30, 60, and 90 degree angles as well as the Pythagorean theorem several



ØISTEIN GJØVIK

Fig. 4 Clever positioning of the axes will help students find the formula for the volume.

times. In addition, it is a good exercise in doing calculations with square roots.

An easier way to find the volume is by placing the shape at the origin of a three-dimensional coordinate system (see **fig. 4**). Using $2s$ as the side length of the original paper, we see that there are six lengths of s on the shape. The height of the pyramid is s, and the triangular base is then calculated to be $s^2/2$. The volume of a pyramid will then be base · height/3 = $(1/3) \cdot (s^2/2) \cdot s$. Because we have two pyramids, the total volume is $s^3/3$. (I rarely observe students or teachers finding the volume this way.)

This problem can be extended by posing questions such as, "If we fill the shape with water, what must the original paper size be if it is to contain exactly 1 gallon?"

FINDING SYMMETRIES

Another way to use this shape in mathematical activity is in finding symmetries. We usually restrict ourselves to looking for rotations and reflections. Students are challenged to verbalize how they think the shape is symmetric. One way is to imagine an axis passing through the apexes of the two pyramids. Because of the regular triangular base, there exist rotational symmetries of 120 and 240 degrees around this axis; we must also include the identity rotation. Another three axes of symmetry can be found in the common plane between the two pyramids. This common plane is a regular triangle, and an axis of symmetry passes through each vertex of this triangle and the midpoint of its opposite side (see **fig. 5**).

One fundamental skill for teachers is finding and reflecting on questions that promote mathematical thinking: "How is this different from … ?" or "In what way are these similar?" Should colors matter when we discuss symmetries, or are we just focusing on shape? Such questions might seem pointless, but we often turn to unusual questions to anticipate what preservice teachers might encounter when they have students of their own.

FINDING COMBINATIONS

Now we organize a parade of shapes, lined up in the classroom. If the shapes are placed with a "front," composed of three faces, toward the audience, as in **figure 1**, how many different fronts can be lined up? If one shape can be made into another by rotation, we will consider them the same; however, if they are the same after a reflection, we will regard them as different. This means that we will consider the order red, green, and blue (clockwise) as a shape different from red, blue, and green (clockwise). In group theory, these two fronts would be considered the same because one can be obtained from the other by a flip in space. The reason that we consider them different shapes is that we will be using them as such in a puzzle activity (discussed later). See Fraleigh (1994, p. 224) for an example of how the same question is handled in group theory.

Students are now given twenty-four copies of **figure 6** (see appendix A, available full size at **www.nctm.org/mt020**). They are asked to use colors to find the total number of different fronts systematically, or they can devise their own systems. We can count the number of fronts in several ways. One is to use the fact that we have a total of $4^3 = 64$ different fronts; however, many of these are the same when rotated. The four fronts with only one color can be ruled out because there will be only one of each. We then have 60 fronts left, and they can all be rotated 120 and 240 degrees to produce the other configurations; hence, we obtain $60/3 = 20$ different fronts with more than one color. In total, we have 24 different fronts on our shapes.

One nice aspect of this activity is that the answer can be found by counting or coloring the different shapes and also by thinking mathematically. This activity also anticipates what preservice teachers may later encounter in their own classrooms.

Being able to make tasks harder or easier is an important skill for teachers to develop. Although it may appear easy, extending a task by doing more than just changing numbers takes practice. The combinations task can be extended by asking, "What will happen with tetrahedrons made of paper of five (or more) different colors?"

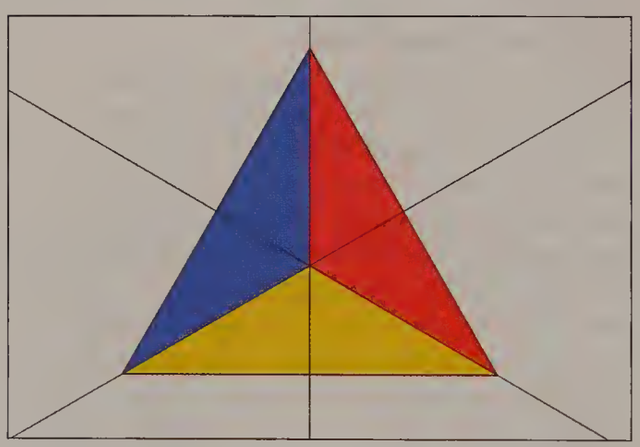


Fig. 5 Axes of symmetry pass through the vertex of each triangle and the midpoint of the opposite side.

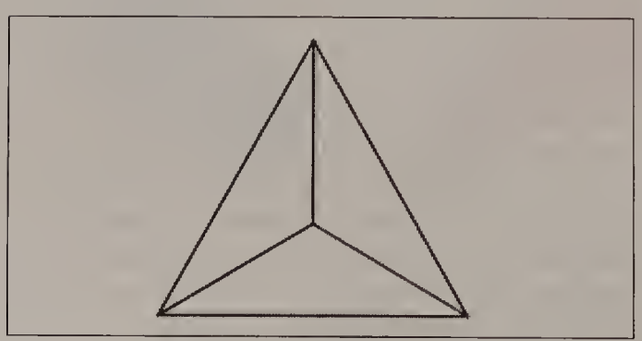


Fig. 6 Students work with this basic shape.

A PUZZLE EMERGES

After discovering the 24 different color combinations of shapes (see appendix B, available full size at **www.nctm.org/mt020**), students cut them out with scissors. We now have pieces for an excellent puzzle originally constructed by the British mathematician Percy Alexander MacMahon (1854–1929). The only two rules are (1) to match each edge of a triangle with one of the same color and (2) to make sure that all the pieces at the rim have the same outer color. One of the more than fifty solutions is shown in **figure 7**. For more on this puzzle, explore the following websites: http://boliaoness.tripod.com/math/24colour.html (solutions); http://www.daviddarling.info/encyclopedia/M/MacMahon_squares.html (extensions); and http://www.gamepuzzles.com/pparlor/mm3.html (a physical version can be obtained from this site).

THE BIRD TETRAHEDRON AS A DIE

On one occasion, some students starting flipping the shape around like a kind of die found in role-playing games. A question that grew out of that activity is whether the bird tetrahedron can be used as a fair die. The fact that two sides, rather than one, face

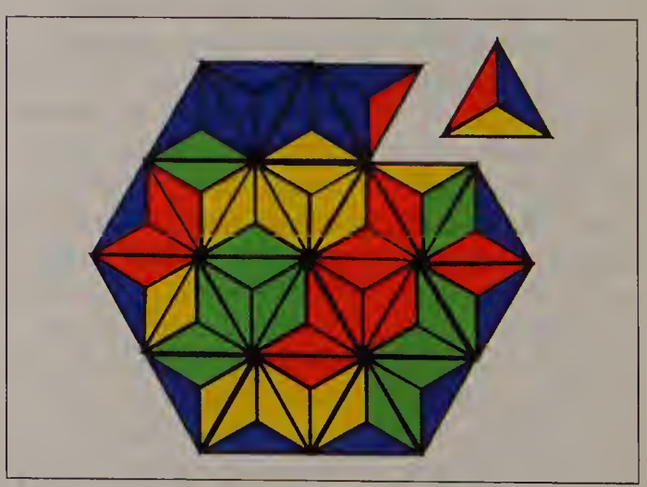


Fig. 7 MacMahon's puzzle challenge involves using the shapes cut out from appendix B.

up presents problems. Students conjectured that the sides could be numbered so that the two sides facing up sum to the numbers 1 through 6. The students started examining different combinations of whole numbers to see what sums can be made. We were not able to solve this task on our first attempt in class; an argument that the task is not possible follows.

Each main side of the shape has three faces. Two of these three face up, and we must consider whether combinations of two natural numbers that give the sums 1, 2, 3, 4, 5 and 6 exist. If one side has three odd numbers (OOO), the sum of any two of these is always even. If one side has three even numbers (EEE), this sum also is always even. If a side is labeled OOE, the sums can be OO or OE, resulting in one even and two odd sums. The last possibility is OEE, and this sum also results in one even and two odd sums. No configuration gives one odd and two even numbers or three odd numbers. From this, we conclude that labeling the sides with natural numbers so that we get three odd and three even numbers all together is impossible.

CONCLUSION

Starting out with some basic origami, we can explore geometry, algebra, combinatorics, puzzles, and dice explorations. As a bonus, students have their own bird tetrahedron and their own set of MacMahon's puzzle pieces to take home. Undoubtedly, readers can find ways of making the bird tetrahedron soar even higher.

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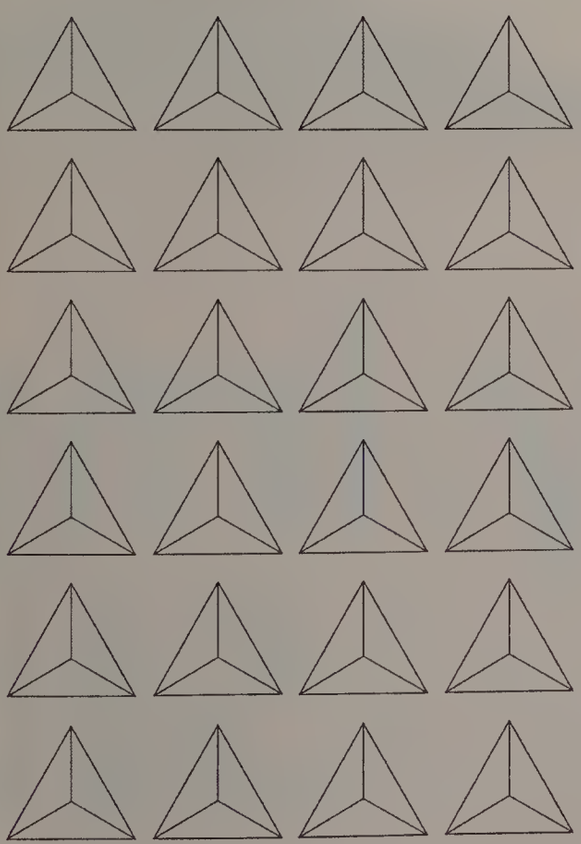
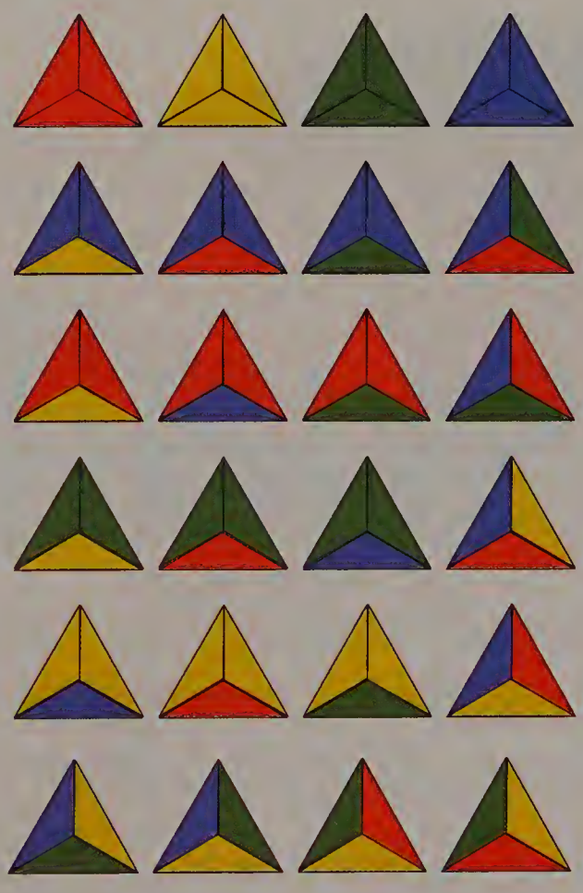
For full-size uncolored and colored figures (appendices A and B), download one of the free apps for your smartphone and then scan this tag to access **www.nctm.org/mt020.**

APPENDIX A



APPENDIX B



Teaching Geometry to Visually Impaired Students

Teaching a visual subject to a visually challenged student inspires strategies that benefit all students in a class.

Christine K. Pritchard and John H. Lamb

aired dents

From Euclid's *Elements* to NCTM's *Focus in High School Mathematics: Reasoning and Sense Making in Geometry* (McCrone et al. 2010), geometry teachers know the important geometric elements that high school students should master, those that enable us to understand the world around us. NCTM (2000) described geometry as "a means of describing, analyzing, and understanding the world and seeing beauty in its structures" (p. 309). Dossey et al. (2002) captured the essence of this aspect of visualization by stating that geometry fosters in students an ability to "visualize and mentally manipulate geometric objects." (p. 200). Clearly, visualization is imperative in understanding geometry.

However, not all students have an inherent or physical ability to look at a figure or drawing and make conjectures on the basis of what they see or visualize the figure's properties. One mathematics teacher was challenged by a cognitively able student who did not have the visual capability to fully "see" the beauty of geometry using her eyes. The teacher's experience follows. (Note: The experiences of the primary author are conveyed in first person, and pseudonyms are used throughout.)

THE NEW STUDENT

The greatest challenge of my teaching career occurred when Jessie joined my geometry class. I had taught geometry for sixteen years and considered myself a fairly competent instructor. I knew my subject and was confident in my ability to help any student make the connections from one geometry topic to another. By the end of each academic year, I had rolled the entire geometry curriculum into a neat bundle ready for each student to carry with him or her to whatever mathematics class or life situation they would encounter next.

Most important, as a geometry teacher, I embraced the necessity of visualizing the subject and emphasized the importance of pictorially representing problem situations. My first question to a befuddled student was always, "Have you drawn a diagram to represent this problem?"

One morning in early August, as I was preparing for the new school year, my school district's teacher for the visually impaired, Mrs. Jones, stopped by my room. She informed me that Jessie, who is completely blind, was going to be in my geometry class.

Teaching the visually impaired was not a completely new experience for me. I had taught many students who required large-print text, and I had even taught entry-level algebra to a blind student several years earlier. But I was completely unprepared to teach a subject so dependent on visual cues to someone who could not see. Almost every problem that my students solve involves making a diagram of some sort and gathering information from that diagram to help find solution strategies. Jessie was in a pre-Advanced Placement class designed to be taught as a college prep class; it consisted of about thirty ninth-grade students. Jessie had a high aptitude in mathematics; however, I was skeptical about her chance for success in this primarily visual subject.

I had only one day to prepare for Jessie. There was no time for me to investigate strategies for teaching geometry to the blind. I simply jumped in and hoped for the best.

As I worked feverishly to set up my room for the arrival of my students the next day, Mrs. Jones dropped off some supplies that Jessie would need for my class. These supplies included a Braille ruler and protractor, a geoboard for graphing, and an audiocalculator. Also included was a board with Mylar® polyester film on it; scratching this board with a plastic "pencil" would raise lines, allowing one to draw crude diagrams.

Although I had some physical tools, I had no instructions on how to teach geometry to a student who could not see. Throughout the school year, as I attempted to teach Jessie, I confronted many impediments. By working toward solutions to these challenges, I began learning new and innovative ways not only to teach this one student but also to improve instruction for all the other students in my classroom.

LOGISTIC CHALLENGES

The Textbook

My district had just adopted a new textbook focused on discovery learning through investigations and problem solving. This textbook had rich examples and descriptions that I had planned to have students read and work through. My initial goal was to have students do this reading silently and then work through the problems in groups. Jessie required a special version of this textbook; unfortunately, the publisher did not yet have a Braille edition available.

Mrs. Jones assured me that the publisher would have the first two Braille chapters by November. However, by November I expected to be well into chapter 5, so the publisher's schedule was not going to do Jessie a whole lot of good. Mrs. Jones was willing to translate a few assignments into Braille but did not have time to translate all the investigative text. As the only secondary school teacher for visually impaired students in the district, she was at that time providing services to eleven children and dividing her time among three campuses.

With no textbook for Jessie to read, I had to find another way for her to be exposed to all the investigative material. I did not want Mrs. Jones just sitting in the back of the class reading the text aloud to her, so I used my favorite teaching strategy—cooperative learning.

I have always placed my students in groups so that they would have a support system while investigating new concepts and working through problems. This approach has been beneficial for all my students, but it was especially helpful for Jessie. The other students in her group did not mind reading the investigations out loud and including Jessie in the discussion. In this way, we worked through the basic concepts of geometry. As always, these cooperative learning groups were helpful for all my students, whether sighted or visually impaired.

Thus, I began learning new ways to teach this one student and improve instruction for all the other students.

Note Taking and Problem Solving

Having the text read aloud was an accommodation beneficial to Jessie. However, hearing the text helped with the more basic concepts of the subject. Jessie, like most other students, needed to be able to take notes, complete scratch work, search these notes, and work while attempting to make sense of the more abstract and difficult topics. She needed to find ways of recording the important aspects of a lesson as well as recording her work when solving problems. To do so required the assistance of Mrs. Jones, a Braille typewriter, and me.

Technological Assistance and Its Limitations

Jessie began the school year with a mechanical braillewriter (a braillewriter is a typewriter that stamps the paper with Braille text). Jessie would type on her braillewriter while other students took notes. By feeling what she had typed, she could understand what she had written, so she could read her notes back when necessary. However, the result was mountains of paper notes that were difficult to organize and use.

Braille paper, similar to construction paper, is larger and thicker than regular paper. When text is converted to Braille, much less fits on a single page. Dick and Kubiak (1997) described this increase in page allocation for Braille: "A pocket dictionary, such as the one printed by American Heritage, can occupy a bookshelf 6 1/2 feet high by 3 feet wide!" (p. 346).

The thickness of paper coupled with the increase in page space needed to write or construct in Braille meant that Jessie's notes tended to create stacks of often-unsearchable material. Jessie could not glance through her notes for a definition or an example as other students could. Her notes for a chapter could fill an entire notebook. Looking up a formula or reviewing a process from a previous problem was an extremely tedious process.

About halfway through the school year, Jessie upgraded to an electronic braillewriter—in actuality, more of a computer than a typewriter. She no longer had to print her notes but could store them on this computer and then recall them when needed. The new braillewriter displayed the notes for her in Braille, so she could open a file and read the notes that she had stored there. Jessie's new braillewriter could also be connected to a printer, so her assignments could be printed in regular text for me and turned in without having to be translated by Mrs. Jones. This new technological tool significantly simplified the note taking process for us all.

The new braillewriter benefited Jessie in numerous ways, but it had limitations too. During the class's study of quadrilaterals, I had integrated problems that involved triangles and quadri-

laterals and that required an understanding of all the properties of both types of figures and how their angles are related. The drawings often took up a whole sheet of standard typewritten paper; using the drawings, students had to find numerous angle measures to solve a given problem. Typically, students wrote each measurement on the drawing as they calculated each one. This scratch work could then be used to find other measurements in the drawing.

This activity presented a huge challenge to Jessie. She could not take the brailled diagram, place it in her braillewriter, and record the measurements in the drawings when she found them and then use them to find other measurements. I began by breaking the problems into smaller pieces and having her work through them a few at a time, but this approach defeated the purpose of her being able to assimilate all the properties of the different shapes and understanding how they fit together.

The solution that we finally came up with made use of the simple thumbtack. Braille paper is more like card stock than regular paper, and the thumbtack worked well for punching small holes in the card stock from the back, thus raising bumps on the front. Mrs. Jones, who often accompanied Jessie to my class, would write the measurements Jessie calculated in the diagram in Braille using the thumbtack, and Jessie could then read the numbers and move forward through the problem. This was a tedious and time-consuming process, but Jessie successfully completed the unit on quadrilaterals.

Symbols

As the mathematics became more complicated, I realized that Jessie was getting lost. The new braillewriter and the hole punching helped her take notes and record scratch work, but much of the language of geometry consists of symbols that allow students to memorize the vocabulary and aid in note taking. Symbols are used to represent lines ($\leftrightarrow$), segments ($—$), and rays ($\rightarrow$) as well as triangles ($\triangle$). Symbols are also used to represent relationships: parallel ($||$), perpendicular ($\perp$), and congruent ($\cong$). The list of symbols is extensive.

Sighted students become accustomed to the symbols as visual clues to use when creating a diagram, labeling parts, or drawing conclusions. When translated into Braille, these symbols take up more space than the words that the symbols represent, so Jessie simply had to memorize the words and their meanings without the benefit of the visual clues that other students enjoyed. For example, when the symbol $\perp$ is used, a student can visually associate the perpendicular lines intersecting to create the "square" 90 degree angles, but Jessie could only use the word "perpendicular" as her clue. Such symbols help students bridge the gap between terminology and concrete representations. Jessie's inability to see or use these symbols created a greater challenge for her in learning the material.

CONTEXTUAL CHALLENGES

One of the toughest challenges was guiding Jessie to an understanding of concepts for which, because of her blindness, she had no context. As mathematics teachers, we often try to connect the abstract topics of the subject with examples of real-life situations to help students better understand them. The available contextual clues for a blind student are greatly decreased from that of a sighted student.

Sighted students become accustomed to the symbols as visual clues to use when creating a diagram, labeling parts, or drawing conclusions.

Perspective

One such contextual issue arose during a lesson using perspective drawing of three-dimensional figures. Jessie needed to feel what she could not see. When we studied polygons, I gave her pattern blocks to feel and trace. When the other students drew tessellations, she built hers using the blocks. Then we began working with three-dimensional solids. Jessie had worked with solids in previous mathematics courses, but, like many students, she seemed to have a hard time equating two-dimensional perspective drawings with the three-dimensional figures that they represented.

Perspective drawings give a two-dimensional account of how a three-dimensional figure is viewed. Jessie had no visual context for this perspective. For example, a *prism* is defined as containing two parallel congruent bases. Often, when a prism is drawn on two-dimensional paper, the drawing looks as if the bases intersect (see **fig. 1**). To make this idea clear to Jessie, we had to trace each face of the prism on the paper and then indicate to her which face of the prism it represented. This was a laborious process. On paper, to indicate perspective, rectangles are sometimes drawn as parallelograms, and circles are sometimes drawn as ovals—another confusing point for Jessie. However, throughout the unit on surface area and volume, the solid figures were always available for her to touch and manipulate.

Solids of Rotation

A final example of Jessie's limited context occurred during the last unit of the school year, in which I

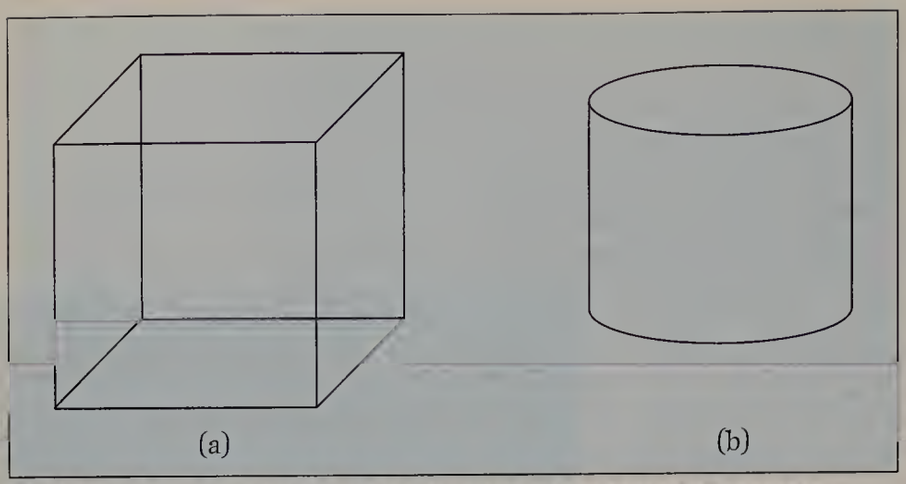


Fig. 1 When drawn on paper, a rectangular prism (**a**) has bases that appear to intersect (the front and back faces, for example). Some of the faces appear to be parallelograms rather than rectangles. This cylinder (**b**) appears to have a base that is shaped like an oval rather than a circle.

combined several topics: graphing lines in the coordinate plane and finding the area and perimeter of the bounded region that they create. Jessie had done most of her graphing on a geoboard with rubber bands, an acceptable process for finding area and perimeter. The last part of this unit involved revolving those bounded regions about an axis and drawing the solid that was created.

This concept, difficult for my sighted students, was overwhelming for Jessie. No amount of verbal explanation from me seemed to help her wrap her brain around this idea, so I had to find a way to make it concrete for her. I needed to come up with some way of constructing a manipulative that would represent this concept.

From card stock, I cut out a right triangle and a rectangle and taped each to the end of a pencil. The pencil was my axis of rotation. When I rotated the pencil, Jessie was able to feel the triangle going around in her hand. She could trace the circular base that the triangle created and understood that the three-dimensional figure swept out was a cone. The rectangle became a cylinder in her hands (see **fig. 2**). Jessie eventually understood that all solids of rotation have circular bases and that she could find the surface area and volume of each. We even managed to revolve a trapezoid to create a frustum.

LOOKING BACK

My experience in teaching Jessie taught me that any student can reach his or her cognitive potential when instruction is tailored to individual needs. I began that school year with unrealistically high expectations of what Jessie could do. In the beginning, I expected her to keep pace with the rest of the class. After all, she was in an advanced class, wasn't she? As I came to understand Jessie's special needs, I learned that reading and understanding problems, drawing diagrams, and recording information all take more time, space, and organizational strategies. I learned that Jessie *could* reach my high expectation of learning the material, just not as quickly as my sighted students.

CHRISTINE K. PRITCHARD

Fig. 2 Affixing a cutout rectangle to a pencil and rotating the pencil gave Jessie a tactile means of understanding volumes of revolution.

I sometimes underestimated Jessie's capabilities. I doubted that a blind student could fully understand a visual subject like geometry. Yes, geometry is a visual subject, but visualization is not reliant solely on one's eyes. Jessie could see the beauty of this subject with her hands. Small bumps on the paper became pictures in Jessie's mind. Thumbtack punctures became scratch work for problem solving. Twirling rectangles became cylinders in Jessie's palms. In essence, Jessie saw mathematics through her "mind's eye" (this phrase is drawn from Fisher and Hartmann's article [2005]).

Each student learns in his or her own way. Every year I differentiate my instruction to meet students' needs. I have used multiple representations, had students act out problems, scheduled silent study time, had cooperative group activities, and tried to understand and implement teaching strategies that catered to all possible learning styles. Through my experience with Jessie, I learned that when instruction was differentiated for her individual needs, she too could learn without the use of her eyes.

Although I did not have time to research the best ways to help Jessie "see" geometry, I believe that she left my class with a solid understanding of the

subject. She was successfully able to complete all geometry objectives during this school year beside her sighted peers. This success I attribute to the persistence of all of us—Jessie, Mrs. Jones, and myself—in finding ways that helped Jessie make sense of the material.

Understanding geometry is not reliant on seeing objects and images with one's eyes. Problems can be represented "visually" in many ways; they can be spoken, read, drawn, and felt. All students can learn; however, they all learn differently.

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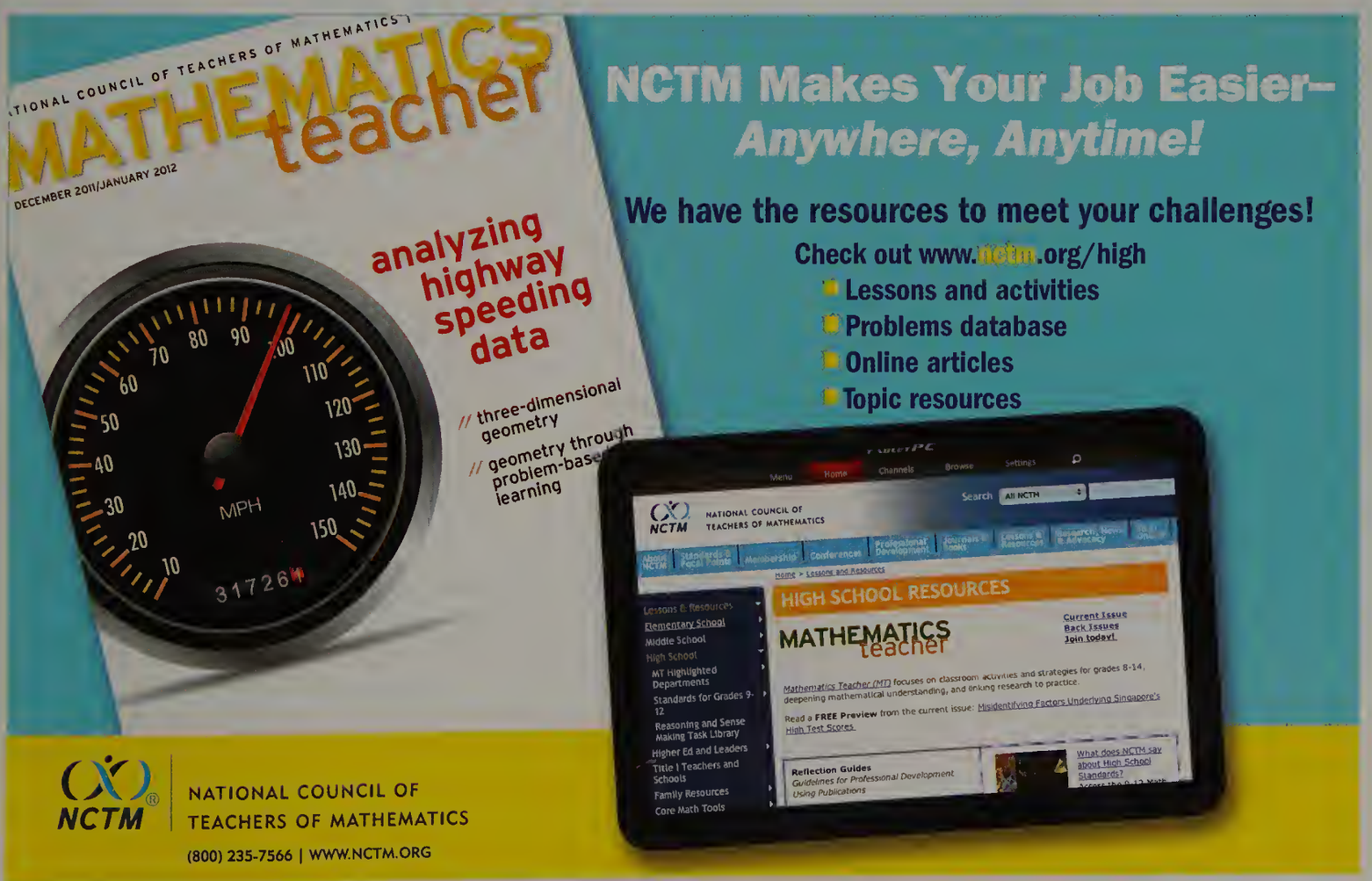


Assessment Can Support Reasoning

Teachers can use data from a research project to enhance their classroom assessment practices.

Christine Suurtamm

"Reasoning and sense making should occur in every classroom every day," states *Focus in High School Mathematics: Reasoning and Sense Making* (NCTM 2009, p. 5). As this book suggests, reasoning can take many forms, including explorations and conjectures as well as explanations and justifications of student thinking. Sense making, on the other hand, is "developing understanding of a situation, context, or concept by connecting it with existing knowledge" (NCTM 2009, p. 4). Classroom assessment can provide opportunities for students to share and enhance their mathematical reasoning and sense making, allowing student reasoning and sense making to become visible to students and teachers so that they can respond in appropriate ways. However, what does such assessment actually look like in classrooms? This article draws on a multiyear research project of grades 7–10 mathematics teachers to describe assessment practices that support students' mathematical reasoning and sense making.

Current thinking and research in classroom assessment promote the use of assessment to support student learning. This is a move away from assessment as merely an event occurring at the end of a unit to provide a mark for reporting and a move toward assessment as a process of ongoing feedback. Such ongoing assessment is referred to variously. I use the term *formative assessment* to refer to the gathering and use of information about students' learning that helps both teachers and students modify teaching and learning activities. Evidence that formative assessment is a powerful lever for improving student learning has been steadily accumulating over the last quarter of a century (Black and Wiliam 1998; Brookhart 2004, 2007; Natriello 1987; Wiliam 2007). In fact, compelling research results indicate that the practice of formative assessment may be the most significant single factor in raising the academic achievement of all students—and especially that of lower-achieving students (Black and Wiliam 1998; Brookhart 2007).

With respect to mathematics, formative assessment provides teachers with a window into students' mathematical reasoning and sense making and creates a forum for student and teacher discussion about such thinking (Gearhart and Saxe

Sense Making

2004; NCTM 1995, 2000). Formative assessment recognizes that mathematical problem solving is a complex task that cannot be measured by merely a paper-and-pencil test; rather, assessment takes on a variety of forms, such as observations, conferencing, questioning, quizzes, progress reports, journal entries, and submission of problem-solving work (Romagnano 2001; Wiliam 2007). Thus, formative assessment prompts the use of a variety of types of assessment to understand and encourage mathematical reasoning and sense making and to guide teachers in their instructional actions. Such assessment also helps create a classroom culture of respect and success (Brookhart 2001; Shepard 2000).

The data from the Curriculum Implementation in Intermediate Math (CIIM) research project help provide some sense of teachers' use of formative assessment and examples of what such assessment might look like in a classroom. The project examined the implementation of a provincewide move toward an inquiry approach to mathematics in Ontario, Canada (Suurtamm and Graves 2007). It examined a variety of teacher practices, beliefs, supports, and challenges through teacher questionnaires ($n = 1096$); analysis of provincial curriculum documents and resources to support the curriculum; focus groups with teachers and leaders in mathematics; and nine case studies of teachers' classroom practice.

What did the data tell us about teachers' assessment practices? A summary of the analysis of the questionnaire data provides an overview of teacher practices. The data from the teacher questionnaire suggest that although many Ontario teachers rely heavily on tests and quizzes, they also use other assessment strategies. The data also suggest that teachers use a greater variety of strategies to get a sense of students' understanding than to determine a grade (Suurtamm, Koch, and Arden 2010). These strategies include performance tasks, conferencing, projects, interviews, observation, and responses of students in class.

Whereas the questionnaire data show that some teachers are incorporating innovative assessment ideas, the case studies provide details of what these look like in classrooms. The classrooms for the case studies were chosen because they promoted mathematical inquiry, investigation, and problem solving.

For each case study, we observed and videotaped four to six mathematics lessons and conducted interviews with the classroom teacher as well as school or department administrators. Examples from three case studies, one each from grades 8, 9 and 10, are presented here to describe assessment practices that supported students' mathematical reasoning and sense making.

CASE STUDY 1: MATH FORUM

Angela is a grade 8 teacher who has established a routine that she calls Math Forum, which she adapted from Fosnot's work (e.g., Fosnot and Dolk 2001). A Math Forum begins with Angela presenting a problem such as this one to her students:

> In the first tug of war, 4 frogs on one side had a tie with 5 fairy godmothers on the other side. In the second tug of war, 1 dragon had a tie with 2 fairy godmothers and 1 frog. The third tug of war was between 1 dragon and 3 fairy godmothers on one side and 4 frogs on the other side. Who won the tug of war?

Angela's rationale for choosing this problem was that it allowed students to use a variety of strategies. Also, she felt that the discussion and comparison of strategies would highlight algebraic reasoning through concepts such as equivalence and substitution used in the various strategies.

After Angela presented the problem, students worked in pairs, writing their ideas and strategies on large sheets of paper. Students had access to a variety of materials, such as linking cubes and two-color counters, and used these to construct mathematical models, make conjectures, and connect their ideas. As students worked, Angela walked around the room, validating and building on students' work while questioning them to develop their ideas and concepts further. She asked student pairs such questions as "Can you elaborate?" or made such suggestions as "This makes sense when you explain it to me. Can you write down your thinking?"

A discussion with one pair of students, Jason and Mike, follows:

Angela: So, Jason, can you explain what you did?
Jason: Well, 4 frogs and 5 fairy godmothers have the same strength. So we divided 4 by 5 to get the strength of 1 frog.
Angela: What made you think of unit rates?
Jason: Because we wanted to get some numbers to compare them.
Angela: Tell me again why you divided by 5?
Mike: Well, we needed some way to get things equal.
Jason: Yeah, so now we have a way to substitute frogs for fairies.

As she observed students working, Angela recorded on a form, next to each student's name, the types of strategies and representations each used. At the end of the forty-minute period, she told the class that the next day some student pairs would present their solutions to the class. In preparation, she reviewed her recorded observations and students' work to determine the pairs that would present and the sequence of the presentations so that the class would see a range of strategies and could connect different representations of similar strategies.

The next class began with the presentation of solutions; students who did not present paraphrased the solutions and asked questions of the presenters. Several pairs had used proportional reasoning. For instance, Pam and Brad used a concept from Pokémon™ and gave the creatures in the story different fighting strengths. They chose the number 20 and "weighted" the strengths of frogs as 5 and godmothers as 4 to balance the first equation. They calculated the strength of the dragon using the second equation and substituted that in the third equation to determine the direction of the inequality (see **fig. 1**).

Some other student pairs also used proportional reasoning in their solutions. For example, Mike and Jason worked with unit rates. Addressing the class, Angela commented, "I'm confused. Why does this work? One group used different numbers than the other group." The class then discussed the concept of proportional reasoning, and one student explained, "It doesn't matter what the numbers are as long as the ratios are the same."

Several other student pairs used different forms of substitution. Josh and Sara used the reasoning shown in **figure 2**, and the class discussed this form of substitution.

Angela then gave students time to try some of the strategies that had been presented. Many used a variation of Josh and Sara's substitution. Some used symbols such as F, G, and D to represent the

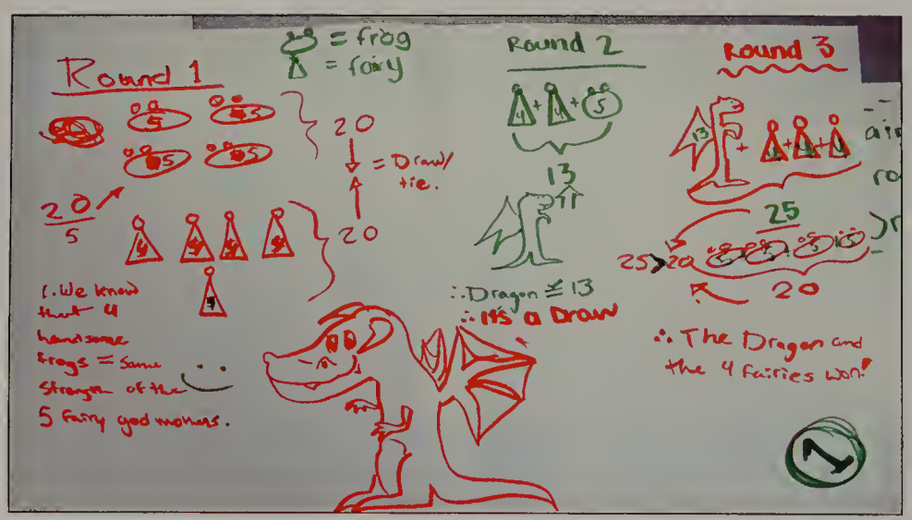
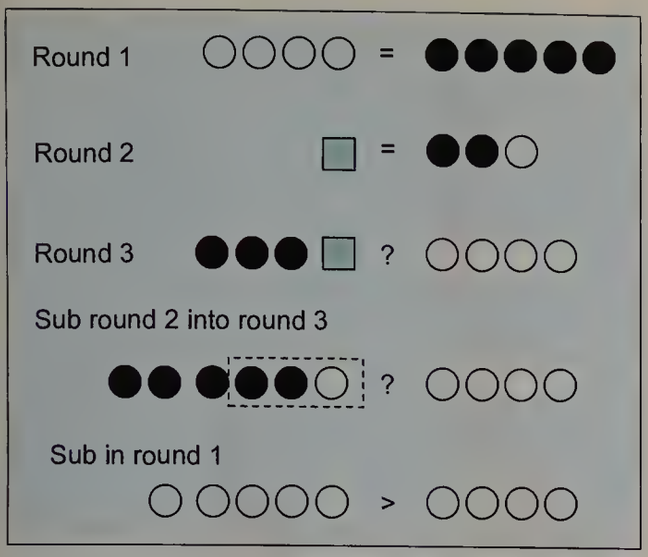


Fig. 1 Pam and Brad's solution made use of Pokémon.



characters; thus, the first round was represented by the equation $4F = 5G$. The class concluded with a discussion of the different ways in which substitution was used and the role of the equals sign as signifying a balance of both sides.

In her interview, Angela stated that Math Forum provides her with multiple assessment opportunities: observing students' collaborative problem solving; reviewing students' written chart-paper solutions; observing the student pairs' presentations; and hearing other students' paraphrasing and questioning. Her observations of the students' work and their reasoning also allowed her to select and sequence the presentations so that students could make connections between the various strategies and concepts. Math Forum gives Angela a strong sense of individual students' as well as the whole class's understanding of mathematical concepts.

CASE STUDY 2: INTERACTIVE QUIZZES

Claire, who teaches grade 9 mathematics, has developed some unique quizzes that play a key role in her assessment plan. She uses both paper-and-pencil quizzes and interactive "clicker" quizzes for formative assessment. (Clickers are a handheld electronic response device.) The quizzes tell both teacher and students of students' progress, Claire reports: "One of the strategies I use is a quiz ... [students] usually get two or three a week for every unit ... they keep their own records, and they do their own assessment of it ... they have a quiz book—it's like their quiz journal."

During paper-and-pencil quizzes, Claire allows students to ask for assistance from her or from peers and to use their notes. She records what her students are able to do with and without support. These notes and the students' performance on the quizzes become the basis for conversations with students about their understanding. Claire states: "Most of them do well on the quiz because they get a lot of support ... then as we get into the unit, I can tell them, 'If you are still asking me for help, you're not going to be ready for your test.' And that's a signal that they have to come in [for help] after school. So I get a lot of mileage out of those quizzes upfront."

Claire also uses clicker quizzes at the beginning of some lessons to generate discussion and get a sense of what students already understand about a topic. In one lesson, Claire was introducing the topic of lines of best fit, moving toward the concept of slope. However, she first wanted to review and consolidate students' sense of correlations so that she could connect these new ideas to their prior knowledge.

Claire began the class by presenting several multiple-choice questions on correlations, posting them one at a time on the interactive whiteboard (see **fig. 3**). Students were given a few minutes to work on the question individually, and then each student chose a response using his or her clicker. A circle graph displaying the range of student responses for the class was then shown on the interactive whiteboard. Student responses also became part of a database that Claire could access later to review individual responses. The circle graph showed that the class had chosen each of the four alternatives about equally.

Students were then encouraged to work with a partner to discuss the question. After the pairs talked, Claire asked students to respond individually to the question again. The responses were now more closely aligned with one another as well as with the correct response. Claire then led a discussion about why that response was correct and why students

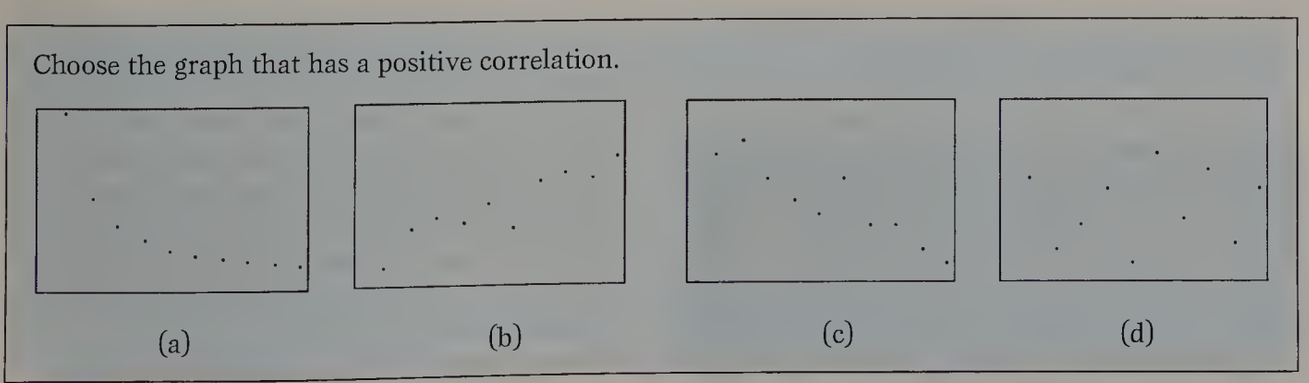


Fig. 3 This multiple-choice question is typical of those on Claire's clicker quizzes.

picked the incorrect responses. Some students stated that they picked solution C first because they were confused about which was positive and which was negative. Many stated that they would rule out D because the data were "all over the place."

In her interview, Claire described this activity as a way to check students' understanding. Although students enjoy using the clickers, occasionally she uses a similar strategy without the clickers: She has students write their response on small individual whiteboards that they hold up so that she can see their responses. She then has them discuss their responses in pairs and respond again. Thus, even without the clickers, Claire gets a sense of students' understanding, and the class still engages in discussion and argumentation, both of which she thinks are vital to developing students' mathematical reasoning.

CASE STUDY 3: QUESTIONING, LISTENING, AND RESPONDING IN WHOLE-CLASS DISCUSSIONS

Terry, a grade 10 teacher, integrates formative assessment within her lessons, particularly in the way in which she questions, listens, and responds to student thinking. This approach is an important component in classrooms in which students are engaged in problem solving and investigation, but it is not always evident in the structure of secondary school mathematics lessons. We observed Terry's grade 10 class over several days as students used their prior knowledge of linear relations to investigate quadratic relations.

Terry began one lesson by modeling a linear relationship with stacked linking cubes. She made a series of stacks of cubes and then drew the associated table of values. The first tower had 4 cubes, the second had 7 cubes, the third had 10 cubes, and the fourth had 13 cubes. She asked, "What kind of relation is this?" After students had offered responses, Terry encouraged them to offer more information about the relationship and their understanding of it by asking questions such as, "Why is it linear?" "How do you know?" and "If you were in grade 3 or grade 4 and I asked, 'Why is this a linear relationship?' and you didn't know anything about slope or first differences or anything, how would you explain it, do you think?"

Students then worked in groups to construct models of a linear function. Terry assigned a first difference to each group, and the groups worked with linking cubes to construct the models. After several minutes, the groups shared their models, and a whole-class discussion of the models ensued. To promote discussion, Terry asked questions such as, "Is there anything else that you notice?" She also prompted students to listen to other students' responses to encourage their reasoning and expand

Fig. 4 In the model of a quadratic relationship, the first and second differences are color coded.

their understanding of particular concepts. Groups then compared models and equations. Two groups, each given a first difference of 3, had different models; one group modeled the equation $y = 3x$ and the other modeled the equation $y = 3x + 4$.

To make the transition to quadratic relations, Terry then presented color-coded linking-cube models of quadratic relationships that she had prepared before class. The color coding helped show the differences between each successive step of the relation and gave students a sense of the curved nature of this relationship. **Figure 4** shows an example of a model of a quadratic relationship and the color-coded first and second differences. Terry then worked with the class to create a table of values for the model, and students determined and discussed the characteristics of the relationship.

Groups were then asked to create a quadratic relationship with specific criteria using different colored blocks. The groups worked in several ways, and Terry asked them to "be prepared to explain how you figured it out." When all the groups had brought their models to the front, Terry used questioning to discuss the models as a class, and then students used graphing calculators to represent their quadratic model graphically. They compared the linking-cube model, the table of values, and the graphical representation for each quadratic. Terry asked, "What does it mean that those two models are equal? What is the same?"

Terry then led a discussion about some characteristics of quadratic relationships. She asked

students whether they noticed any patterns between the equations and the first and second differences of each table. Several students recognized that the coefficient of x^2 appeared to be half the second difference. At least one student also noticed that the way in which the graph opened depended on the sign of the coefficient of x^2. As the class ended, Terry asked students to come up with some conjectures about other properties of quadratics and to be prepared to share them the next day.

Throughout this lesson, Terry encouraged students not only to consider what they were doing and how they were doing it but also, by thinking about what makes a relation linear or quadratic, to question why. Terry claimed that her questioning provided feedback to the students; in addition, she learned a great deal about her students' prior knowledge, misconceptions, and current understanding of mathematical ideas. Terry also used students' responses to adjust her pedagogical strategy.

CONCLUSION

Incorporating new and varied forms of assessment helps make mathematical reasoning and sense making visible to both students and teachers. Walking around the room while students solved problems helped Angela see students' reasoning and how they connected mathematical ideas. It also helped inform her selection and sequencing of the presentation of students' solutions. These presentations made the students' varied strategies and reasoning visible to the entire group, allowing students to make connections to and between the various mathematical ideas presented. Claire's use of clicker quizzes and discussion of responses brought student reasoning to the forefront. Terry's questions sought more than just correct answers; they prompted students to reason and make mathematical sense about the similarities and differences between relationships.

These three teachers' practices provide clear examples of ways to incorporate formative assessment both to generate and view students' mathematical reasoning and sense making and to inform next steps.

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FIFTEEN:

Combining Magic Squares and Tic-Tac-Toe

The game Fifteen can motivate students to develop problem-solving skills and mathematical thinking.

Joseph B. W. Yeo

The game Fifteen immediately engages students. Let's jump right in.

> Nine disks marked with the digits 1 through 9 are placed faceup on the table. Two players take turns picking one disk at a time. The winner is the first player to obtain the sum of exactly 15 from among any three of his or her disks. (Adapted from Mason, Burton, and Stacey 1985)

I start by picking 8. Which number will you choose? If you did not select 5, I will definitely win. For example, you pick 4. Then I choose 6. You have to pick 1 to prevent me from winning with 8 + 6 + 1 = 15. Then I choose 2. Now I can win by picking 5 or 7 because 8 + 2 + 5 = 15 and 6 + 2 + 7 = 15. You cannot win in the next step (because you only have 4 and 1), so I will win regardless of whether you choose 5 or 7. To know which numbers to pick to win, see **figure 1**.

Playing Fifteen is like playing tic-tac-toe on a 3×3 magic square in which the three numbers in each row, column, and diagonal add up to 15. The game can be used in the classroom to motivate and help students learn mathematics and develop heuristic skills such as examining all possible scenarios and systematic listing (Pólya 1957), spatial visualization (Presmeg 2008), and thinking skills, including predicting, conjecturing, generalizing, and checking (Ainley 1990).

A REVIEW OF MAGIC SQUARES

When my students finally realized that playing Fifteen was like playing tic-tac-toe on a magic square, they usually felt that this was enough to give them command of the game. Students may need to be guided to explore other possibilities, however.

For example, if a player can win by choosing three numbers as shown by the Xs in **figure 2**, then playing Fifteen will be quite different from playing tic-tac-toe on a magic square. Thus, students need to prove that the only combinations of three numbers that sum to 15 are those that lie in each row, column, and diagonal of the magic square.

One method that some students have found to determine all possible sums is to start with 1 and divide the sum of the other two numbers (i.e., 14) by 2 to give 7. Because 7 + 7 is impossible, the

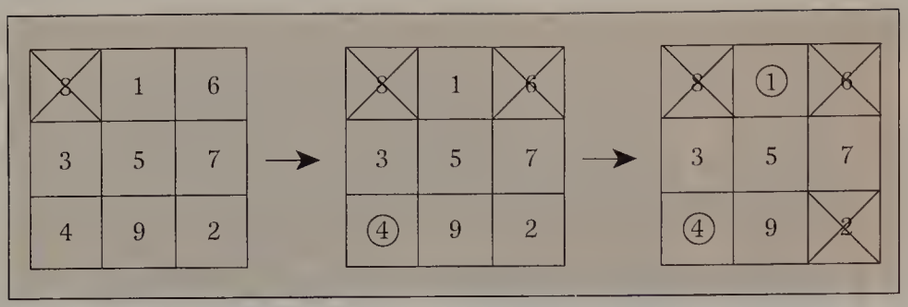

Fig. 1 Choosing 8 as a first move will enable the first player to win a game of Fifteen.

next logical choice is 6 + 8 (decrease the first number by 1 and increase the second number by 1), followed by 5 + 9 (repeat the same process), and the impossible result, 4 + 10. At this point, we are sure that there are only two possibilities if we start with 1.

Using the same method with 2, 3, and 4 as the starting number, we can sieve out all the eight combinations, as follows:

$$\begin{aligned} 15 &= 1 + 6 + 8 \\ &= 1 + 5 + 9 \\ &= 2 + 6 + 7 \\ &= 2 + 5 + 8 \\ &= 2 + 4 + 9 \\ &= 3 + 5 + 7 \\ &= 3 + 4 + 8 \\ &= 4 + 5 + 6 \end{aligned}$$

The next starting number is 5. Because 5 + 5 + 5 is not possible, if we use the same process as earlier, one of the last two numbers will be less than 5—for example, 5 + 4 + 6—but we would have found this sum using 4 as the starting number. So there are no more new combinations if the starting number is 5. Many students were tempted to stop here, but they need to reason that if the starting number is larger than 5, then all the sums obtained will be repetitions.

The next step is to construct a magic square. We will discuss a method in which students can reason through the problem instead of memorizing procedures.

Students usually remember that 5 is in the center square, probably because 5 is the average, or "middle," of the nine numbers. The important question is whether all the corners of the magic square should be occupied by odd or even numbers. Students can reason this way: If the corners are filled by the remaining odd numbers (see **fig. 3**), then the first row (or column) will show that the arrangement is impossible because odd + even + odd is even but the sum 15 is odd. Therefore, all even numbers must be in the corners.

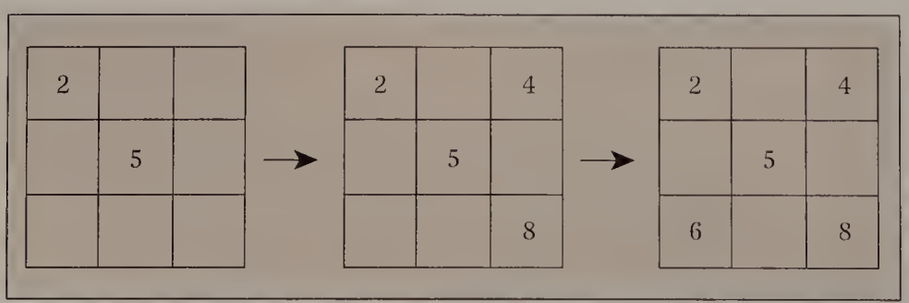

Fig. 4 This magic square is oriented differently (it is rotated through 180°) from the one shown in **figure 1**. In fact, all eight 3 × 3 magic squares can be obtained from one another by rotation or reflection.

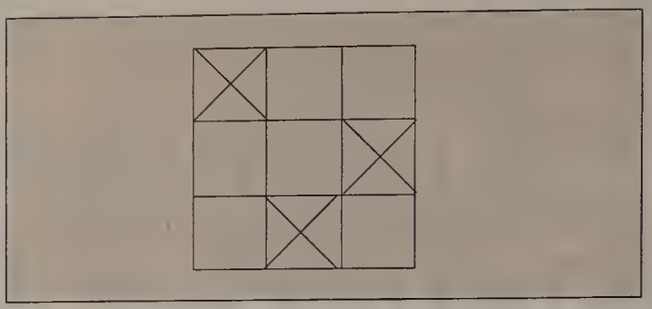

Fig. 2 Can a player get a sum of 15 with three markers that are not in the same row, column, or diagonal?

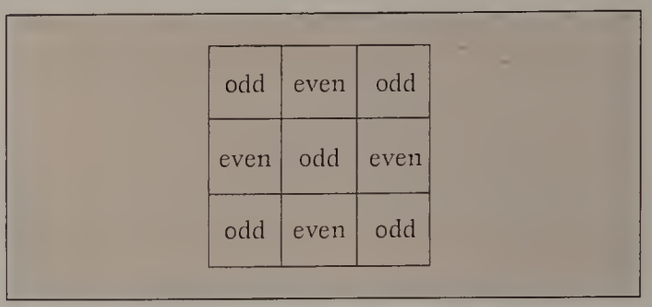

Fig. 3 Placing odd numbers in the corners of a magic square cannot result in a sum of 15.

Without loss of generality, we can put 2 in any corner (see **fig. 4**). The opposite corner must be 8 because 2 + 5 + 8 = 15. Without loss of generality, we can put 4 in either remaining corner. The last corner must be 6 because 4 + 5 + 6 = 15. The remaining squares can then be filled with the remaining odd numbers.

We next examine how students can win at tic-tac-toe, a skill that will help them strategize when playing Fifteen.

WINNING STRATEGIES FOR TIC-TAC-TOE

Many students do not understand the concept of a winning strategy. When Civil (2002) asked students to find a winning strategy for the game Nim, the strategies that students found were often dependent on some particular moves by the opponent; if the opponent made a different move, the strategy might not work.

Moreover, the term *winning strategy* has different meanings. In Nim, a winning strategy that ensures victory is possible (the only criterion being whether the player should start first). This is a sure-win strategy. But in tic-tac-toe, there are no sure-win strategies; a winning strategy is simply one that will maximize the chance of a player winning. This will be my definition of a *winning strat-*

egy. Some winning strategies for tic-tac-toe follow.

If player 1 starts with the center square and each player makes optimal moves, the game will always end in a draw.

If player 1 places an X in a corner, player 2 must place an O in the center; otherwise, player 2 will lose. For example, if player 2 puts the first O in the corner opposite the first X (see **fig. 5**), player 1 should place an X in any remaining corner, thus forcing player 2 to put an O in between the two Xs to prevent player 1 from winning. Player 1 can then win by placing an X in the last corner. Readers can figure out how to win the game if player 2 has chosen to put the first O somewhere other than the center square.

Therefore, if player 1 starts at a corner, player 2 will lose unless he or she puts an O in the center. If player 2 does not know this strategy and is equally likely (which may not be true) to put an O in any one of the remaining eight squares, then the probability of player 1 winning is 7/8 = .875.

But this probability is actually higher because player 1 can still win if player 2 puts the first O in the center (see **fig. 6**). Player 1 should place the second X at the corner opposite the first X and hope that player 2 puts the second O at a remaining corner. Then player 1 will win by placing the third X in the last corner. Readers can show that the game will be a draw if player 2 puts the second O elsewhere.

Therefore, if player 2 puts the first O in the center and is equally likely to place the second O in any one of the remaining six squares, then the probability of player 1 winning is 2/6 = 1/3. Hence, if player 1 begins by placing X at a corner, then the probability of him or her winning is

$$\frac{7}{8}+\frac{1}{8}\cdot\frac{1}{3}=\frac{11}{12}\approx .917.$$

However, player 2 is more likely to put the first O in the center if player 1 starts at a corner, so the assumption that player 2 is equally likely to put an O in any square is invalid. Thus, if player 1 starts at a corner, the probability of winning is actually less than .917, but it is still better than starting at the center, which will usually lead to a draw.

If player 1 starts with an edge-square, the game is wide open. Either player can win, although the game will most often end in a draw. Therefore, a winning strategy for player 1 is to first place an X at a corner.

Player 2 needs to know how to maximize his or her chances of winning or what must be done to prevent a loss. If player 1 starts at a corner, player 2 (as discussed previously) can prevent him or her from winning. But if player 1 starts with an edge-square, there are many cases. We will discuss only two.

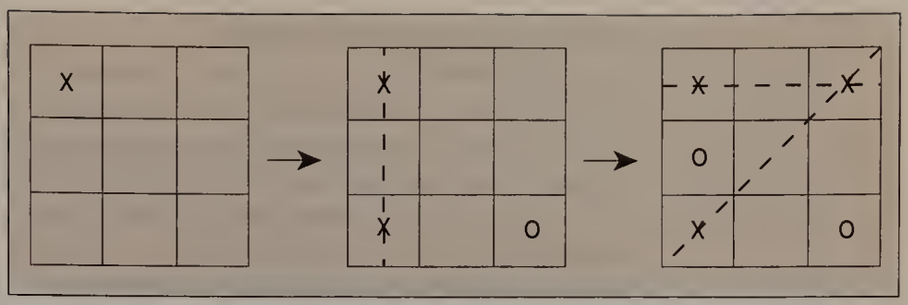

Fig. 5 If player 2's first move is not in the center, player 1 will always win.

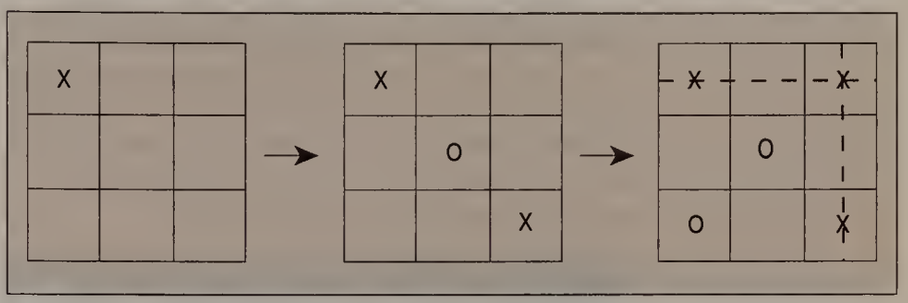

Fig. 6 If player 2's first move is to place O in the center square, player 1 can still win.

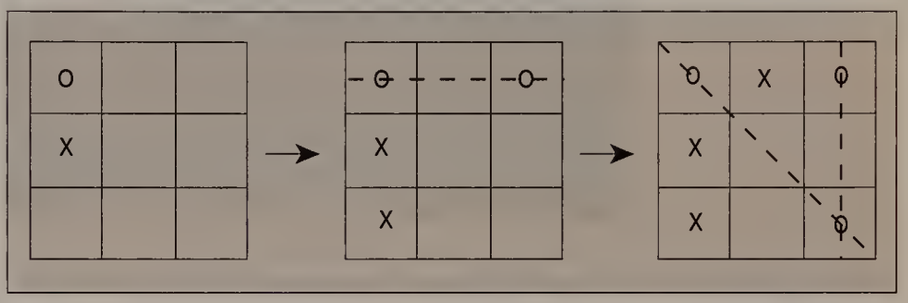

Fig. 7 If player 1 does not place his first X at a corner or at the center, it is possible (but not likely) that player 2 will win.

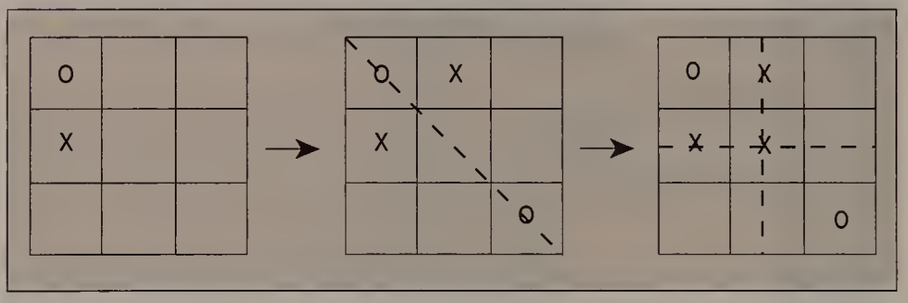

Fig. 8 In this scenario, player 2 loses.

Figure 7 shows player 1 starting with an X on an edge-square and player 2 then placing an O on a corner next to the X. If player 1 puts the second X in the other corner next to the first X, then player 2 can win as shown.

Figure 8 shows the same start as **figure 7**. But if player 1 chooses to put the second X as shown in **figure 8**, then player 2 must not place the second O in any remaining corner; otherwise, he or she will lose when player 1 puts the third X in the center. If player 2 is equally likely to place a second O in any one of the six remaining squares, then the probability that he or she loses at this stage is 3/6 = .5.

STRATEGIZING FOR FIFTEEN

Playing Fifteen is more difficult than playing tic-tac-toe because players must visualize the magic square; otherwise, opponents will be aware of the strategy. Two examples illustrate this difficulty.

Example 1: Suppose that player 1 picks 3. Then player 2 chooses 8, and player 1 selects 1. Which numbers should player 2 *not* pick? It may take player 2 some time to figure out that she should not choose 2, 4, or 6; otherwise, player 1 can pick 5 and then win by choosing 7 or 9 because $3 + 5 + 7 = 15$ and $1 + 5 + 9 = 15$. But if player 1 plays out these moves on the magic square in the orientation shown in **figure 1**, she will see more clearly why she should not pick 2, 4, or 6 (this strategy is the same as the tic-tac-toe strategy in **fig. 8**).

Example 2: Suppose that player 1 picks 9. Then player 2 chooses 4, and player 1 selects 3. Which number should player 2 *not* pick? Again, it takes some time to reason that she should not choose 2, 6 or 8; otherwise, player 1 can pick 5 and then win by choosing 1 or 7 because $9 + 5 + 1 = 15$ and $3 + 5 + 7 = 15$.

Both examples use exactly the same tic-tac-toe strategy as shown in **figure 8**. However, example 1 uses the magic square in the orientation shown in **figure 1**, whereas example 2 uses the magic square in the orientation shown in **figure 9**.

Because the magic square has eight orientations, the same tic-tac-toe strategy illustrated in **figure 8** will translate to eight different scenarios in Fifteen. For this reason, Fifteen is a more difficult but more interesting game to play than tic-tac-toe. Moreover, there are many possibilities if player 1 starts with an odd number other than 5.

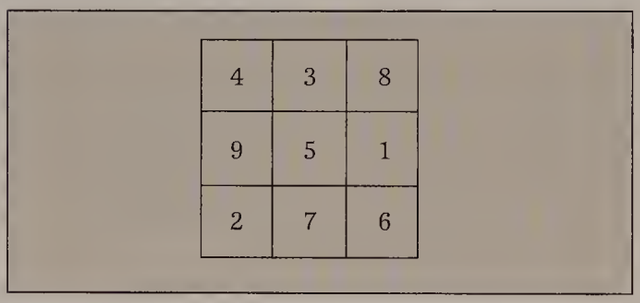

Fig. 9 If player 1 chooses 9 and then 3 after player 2 chooses 4, where should player 2 *not* play?

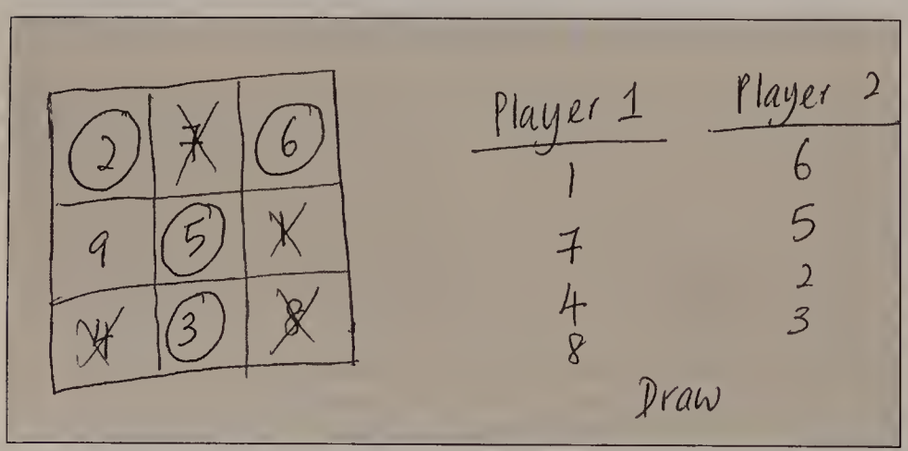

Fig. 10 After player 2 has chosen the number 5, the game will always end in a draw.

Figure 10 shows the work of a student who was investigating the same scenario as shown in **figure 8** but with the magic square in yet another orientation. This student was not able to find a strategy because she kept playing the game to the end without realizing that after a critical move, player 2 will always lose or draw. The critical move in this scenario is what player 2 will choose at the second move.

Many students are unable to move away from the details to gain an overview of the game. For some, it may be easier to investigate a winning strategy for tic-tac-toe first (without all the numbers confusing them) and then apply it to Fifteen than to investigate a winning strategy for Fifteen by directly playing tic-tac-toe on a magic square.

Some scenarios that illustrate the main winning strategies for playing Fifteen follow. Try to investigate them without looking at a magic square first.

1. If you play first, should you choose an even or an odd number?
2. If your opponent plays first and picks an even number, what number must you choose to avoid a loss?
3. Suppose that you play first and pick 8. Your opponent then chooses 3. What are the three numbers that you can choose to ensure a win?
4. Suppose that your opponent plays first and picks 6. Then you choose 5, and your opponent picks 4. Which two numbers should you *not* pick?
5. Suppose that your opponent plays first and picks 7. Then you choose 2, and your opponent picks 9. Which three numbers should you *not* pick?

AN EXTENSION

One way to extend this game to ensure a sure-win strategy follows:

> Nine disks marked with the digits 1 through 9 are placed faceup on the table. Two players take turns picking one disk at a time. The winner is the first player to obtain the sum of exactly 15 from among any *two or more* of his or her disks.

Because it is possible to win by using two disks, there are two more winning combinations: $6 + 9 = 15$ and $7 + 8 = 15$. So player 1 will definitely win if he chooses 6 or 8 because player 2 has to pick 9 or 7, respectively, to prevent player 1 from winning. But, as we have discussed, player 2 must choose 5 on her first move to avoid losing if player 1 starts with an even number. Therefore, this extension has a sure-win strategy for player 1.

CONCLUSION

Most students love to play games. Ernest (1986) believed that games could be used to teach mathematics effectively in four areas: motivation, concept development, reinforcement of skills, and practice of problem-solving strategies. Fifteen is an interesting and thought-provoking game that helps students learn mathematics at the same time.

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SEPTEMBER

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"Thirty days hath September...." The answers to this month's Calendar are the integers from 1 through 30. Each integer occurs exactly once.

If

$$\sqrt{144} + \sqrt[3]{216} = \sqrt{a} = 6\sqrt{b},$$

find the value of

$$\frac{a}{4b}.$$

1

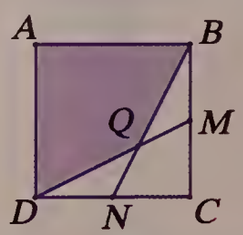
The sum of two positive integers is 4 times the smaller. The positive difference of their squares is 8 times the larger. What is the product of the two integers?

2

Nick has two part-time jobs that pay him by the hour. When he works twice as many hours at his day job as at his night job, his weekly pay is 4/5 what he would earn if he reversed the number of hours worked at the two jobs. How many hours must Nick work at his day job to earn what he would earn in 10 hours at his night job?

3

Square $ABCD$ is 6 in. on a side. M and N are the midpoints of $\overline{BC}$ and $\overline{DC}$, respectively, and $\overline{BN}$ and $\overline{DM}$ intersect at Q. What is the area of quadrilateral $ABQD$?



4

A right triangle and a square are arranged as shown in the left figure. The square hides 80% of the triangle's area. When the layers are reversed, as shown in the right figure, only 15% of the square's area is visible. If the square measures 4×4 cm, what is the triangle's area?

5

Grover, Bert, and Ernie entered a jelly bean–guessing contest in which the winning guess is the closest guess that does not exceed the actual count. Grover won the contest; his guess was twice as far from the actual count as Ernie's guess. Bert's guess of 782 cut Ernie's error in half. The sum of their guesses was 2278. Find the sum of the digits of the actual number of jelly beans.

6

Points Q (10, 8) and R (24, 10) are the endpoints of a diameter of a circle centered at P. Line l is tangent to the circle at S (12, 14). What is the y-intercept of line l?

7

Call a two-digit number A. Reverse the digits of A to create a second two-digit number, B. Find the sum of the digits of the largest value for A such that

$$A : B = 4 : 7.$$

8

If 1/5 and 7/15 are the first and fifth terms of an arithmetic sequence, what is the sum of the second, third, and fourth terms of the sequence?

9

A positive integer is divided by 2, 3, 4, and 5, and the remainders are added together. Find the sum of the digits of the smallest integer that makes the sum as large as possible.

10

Point P lies on the perpendicular bisector of segment ST. If S has coordinates (5, 14), T has coordinates (–3, 2), and P has coordinates $(-a/2, a)$, what is the value of a?

11

Find the number of ordered pairs of positive integers (x, y) that satisfy the following equation:

$$x^y = 2^{64}$$

12

In how many ways can we select two different integers from the set

$$\{12, 13, 14, \ldots, 20\}$$

such that their sum is even?

13

Find the area of the region bounded by the positive x- and y-axes and the graph of

$$y = -3|x - 1| + 9.$$

14

The numbers a, b, c, d, and e are the first five terms in a sequence. Each term beginning with c is the sum of the previous two terms. If

$$a = e = 9,$$

find the value of d.

15

Gino is thrilled with his new pizza cutter, which he tests on a freshly baked pizza. He makes 6 straight cuts across the round pizza, and no 3 cuts are concurrent (i.e., they do not intersect at a single point). What is the maximum number of pieces of pizza that Gino's cuts have created?

16

For five consecutive integers, the sum of the even integers is 60. What is the value of the smallest odd integer?

17

The French mathematician Adrien-Marie Legendre, who was born on September 18, 1752, first proved in 1794 that π^2 is an irrational number. Find a rational approximation of π^2, correct to four decimal places, and give the sum of the first four digits to the right of the decimal. (You may use your calculator.)

18

Find the sum of the solutions to

$$|x - 14| = 0.25|x - 21| + 2.$$

19

How many positive integers between 2012_7 (2012 is written in base 7) and 3012_7 are divisible by 20_7?

20

Let A represent the number of combinations of three different integers selected from the set

$$\{12, 13, 14, \ldots, 20\}$$

such that the sum of the integers is even. Find the sum of the digits of A.

21

For how many integers will the square of the integer decreased by twice the integer produce a difference of less than 4?

22

Let S_n = the sum of the squares of the first n natural numbers, and let Q_n = the square of the sum of the first n natural numbers. Find the smallest value for n such that

$$Q_n - S_n > 2012.$$

23

The integers 220, 251, and 304 represent three consecutive perfect squares in base b. Determine the value of b.

24

Consider all the possible combinations of 3 coins using only pennies, nickels, dimes, and quarters. The total value of these combinations exceeds \$6.00 by k cents. What is the value of k?

25

The dimensions of a right rectangular prism are in arithmetic progression, with lateral edge greater than either base edge. The sum of the areas of the two bases is 5 ft.^{2}, and the prism's volume is 10 ft.^{3}. Find the lateral area of the prism.

26

The 9-digit number

$$8544a5894$$

is divisible by 7. What is the sum of the possible values for a?

27

If

$$f(x) = 3x \text{ and } g(x) = 1/x - 3,$$

find the value of

$$\left[f^{-1}\left(g^{-1}(-1)\right)\right]^{-1}.$$

28

In how many different ways can we select three different numbers from the set

$$\{12, 13, 14, \ldots, 20\}$$

such that the sum is divisible by 3?

29

Triangle ABC has vertices A (0, –5), B (–7, –6), and C (–3, –2). $\triangle A'B''C''$, the image of $\triangle ABC$, is the result of two transformations—reflection in the y-axis followed by a 90° (counterclockwise) rotation about the origin. Find the sum of the x- and y-coordinates of the vertices of $\triangle A'B''C''$.

30

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SOLUTIONS to calendar

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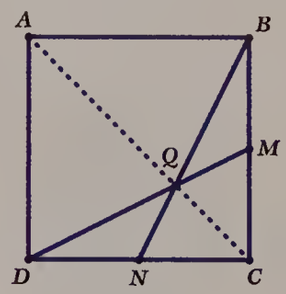
1. 9. We have the following: $\sqrt{144} + \sqrt[3]{216} = 12 + 6 = 18$. Therefore, $a = 18^2 = 324$. We know that $6\sqrt{b} = 18 \rightarrow \sqrt{b} = 3 \rightarrow b = 9$. Then $a/(4b) = 324/36 = 9$.

2. 27. Let a and b be the two integers, $a < b$. Then $a + b = 4a$, and $b^2 - a^2 = 8b$. The first equation implies that $b = 3a$, so substitute $3a$ for b in the second equation: $9a^2 - a^2 = 8(3a) \rightarrow a^2 = 3a \rightarrow a = 3$ or $a = 0$. Discard 0 since $a, b > 0$. The larger integer is $3(3) = 9$, and the product of the two integers is 27.

3. 20. The ratio of the hourly rates is a constant, regardless of the number of hours worked. Let d = Nick's hourly rate at his day job, and let n = Nick's hourly rate at his night job. Then $2d + n = (4/5) \cdot (2n + d) \rightarrow 2d + n = (8/5)n + (4/5)d \rightarrow (6/5)d = (3/5)n \rightarrow 2d = n$. Since Nick's hourly pay at his night job is twice that of his day job, he must work 20 hours at his day job to match his pay for 10 hours at his night job. (This is a Labor Day problem.)

4. 24. Draw diagonal $\overline{AC}$. Q lies on $\overline{AC}$ by symmetry. $\triangle BCN$, with height 6 in. and base 3 in., has area 9 in.^{2}. Median QM divides $\triangle QBC$ into two smaller triangles of equal area. Since $\triangle NQC$ is the reflection of $\triangle MQC$ in $\overline{AC}$, these two triangles are equal in area. We conclude that each of $\triangle DQN$, $\triangle NQC$, $\triangle MQC$, and $\triangle BQM$ has area 3 in.^{2}, leaving $36 - 4(3) = 24$ in.^{2} for quadrilateral $ABQD$.



5. 17. The square has area 16 cm^2. When the triangle covers the square and only 15% of the square is visible, 85% of the square's area, or 13.6 cm^2, is hidden. When the square covers the triangle, 13.6 cm^2, or 80%, of the triangle's area is hidden. The triangle's area is $13.6/0.8 = 17$ cm^2.

6. 18. If Grover won the contest, then both Bert's guess and Ernie's guess exceeded the actual count. Let c = the actual count, and let x = the amount by which Bert's guess exceeded the actual count. Then Bert's guess is $c + x = 782$, Ernie's guess is $c + 2x = 782 + x$, and Grover's guess is $c - 4x = 782 - 5x$. We have $782 + 782 + x + 782 - 5x = 2346 - 4x = 2278 \rightarrow x = 17$. The actual number of jelly beans is $782 - 17 = 765$, and the sum of the digits is 18.

7. 2. Since Q and R are the endpoints of a diameter, the center of the circle, P, has coordinates $((24 + 10)/2, (10 + 8)/2) = (17, 9)$. The radius drawn to the point of tangency (12, 14) has slope $(9 - 14)/(17 - 12) = -1$. The negative reciprocal of -1 is 1, so line l has slope 1 and contains (12, 14). An equation for l is $y - 14 = 1(x - 12) \rightarrow y = x + 2$. Line l has a y-intercept of 2.

8. 12. Assign t to the tens digit and u to the units digit of A. Then

$$\frac{10t + u}{10u + t} = \frac{4}{7} \rightarrow$$
$$70t + 7u = 40u + 4t$$
$$66t = 33u$$
$$\frac{t}{u} = \frac{1}{2}.$$

This result implies that A can have any of the values 12, 24, 36, and 48, of which 48 is the largest. The sum of the digits is 12.

9. 1. If d is the common difference between successive terms of the sequence, then the second, third, and fourth terms can be represented as $1/5 + d$, $1/5 + 2d$, and $1/5 + 3d$. We know that $1/5 + 4d = 7/15 \rightarrow 4d = 4/15 \rightarrow d = 1/15$. The sum of the three terms is $3(1/5) + 6(1/15) = 1$. Changing 1/5 to 3/15 makes the solution easier to see.

10. 14. The largest remainder when an integer is divided by a divisor d is $d - 1$. So the largest sum of remainders after division by 1, 2, 3, 4, and 5 is $0 + 1 + 2 + 3 + 4 = 10$. To obtain these remainders, we need an integer that is 1 less than a multiple of 2 *and* 1 less than a multiple of 3 *and* 1 less than a multiple of 4 *and* 1 less than a multiple of 5. This integer is simply 1 less than the least common multiple of $\{1, 2, 3, 4, 5\}$, which is $60 - 1 = 59$. The sum of the digits of 59 is 14. (To solve this problem, we do not really need to know that the sum of the remainders is 10.)

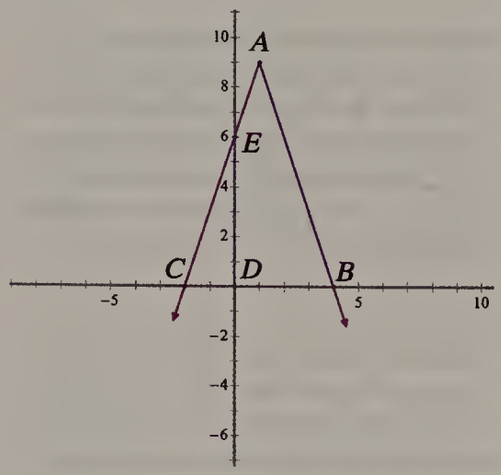
11. 13. The slope of $\overline{ST}$ is $12/8 = 3/2$. The midpoint of $\overline{ST}$ is $(1, 8)$. So the perpendicular bisector of $\overline{ST}$ has an equation: $y - 8 = (-2/3)(x - 1)$. Substitute $-a/2$ and a for x and y, respectively: $a - 8 = (-2/3)(-a/2 - 1) \rightarrow 3a - 24 = a + 2 \rightarrow a = 13$.

Alternate solution: $PS = PT$, so the distance formula can be applied.

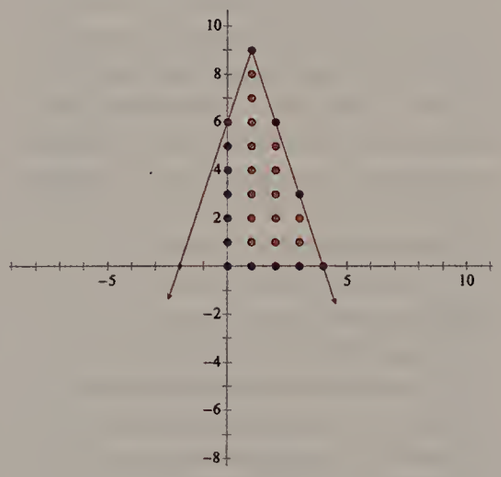
12. 7. The number of ordered pairs depends on the number of positive factors of 64. There are seven such factors: 1, 2, 4, 8, 16, 32, and 64. If $x = 2^{64}$, then $y = 1$. If $x = 2^{32}$, then $y = 2$. If $x = 2^{16}$, then $y = 4$. The remaining ordered pairs are (256, 8), (16, 16), (4, 32), and (2, 64).

13. 16. The given set of integers contains five even integers and four odd integers. An even sum requires two even addends or two odd addends. There are ${}_5C_2 = 10$ ways to choose two even integers and ${}_4C_2 = 6$ ways to choose two odd integers. Thus, we have $10 + 6 = 16$.

14. 21. The V-shaped absolute value graph opens down. Its vertex is $(1, 9)$. The y-intercept is 6. The x-intercepts are found at $|x - 1| = 3 \rightarrow x = 4$ or $x = -2$. The graph shows that the region in quadrant I is a quadrilateral. Its area can be found by dividing it into a trapezoid and a triangle (in two different ways) or by subtracting the area of $\triangle CDE$ from that of $\triangle ABC$. Using the last method, we have area $\triangle ABC$ – area $\triangle CDE = (1/2)6 \cdot 9 - (1/2)2 \cdot 6 = 27 - 6 = 21$.



Alternate solution: Use Pick's theorem: The number of interior lattice points plus half the number of boundary points minus $1 = 15 + 0.5(14) - 1 = 21$.



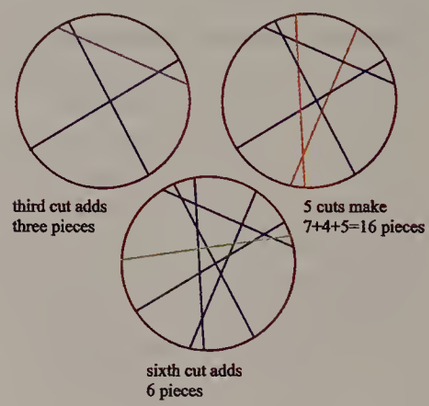
15. 3. Beginning with c, each term is the sum of its two immediate predecessors, so $c = a + b$ and $d = b + c = a + 2b$. Then $e = c + d = (a + b) + (a + 2b) = 2a + 3b = 9$. Since $a = 9$, $b = -3$ and $c = 9 + (-3) = 6 \rightarrow d = 3$.

Alternate solution: Beginning with c, each term is the sum of the previous two terms, yielding the following:

$$9 + b = c$$
$$b + c = d$$
$$c + d = 9$$

We can add c to both sides of the second equation to obtain $b + 2c = c + d$, implying that $b + 2c = 9$. Substitute the first equation into this one to obtain $b + 2(9 + b) = 9 \rightarrow b = -3$. Thus, $c = 6$ and $d = 3$.

16. 22. If Gino intersects each previous cut once as he makes each new cut, he makes a new piece when he reaches a previous cut or the edge of the pizza. It is easy to see that 1 cut makes 2 pieces and that 2 cuts make 4 pieces. The third cut adds 3 pieces, for a total of 7 pieces. The fourth cut adds 4 pieces, the fifth cut adds 5 pieces, and the sixth cut adds 6 pieces, for a total of 22 pieces. The terms of the sequence increase by consecutive counting numbers, and 2 is the first term; thus, these "pizza" numbers differ from the triangular numbers by 1.

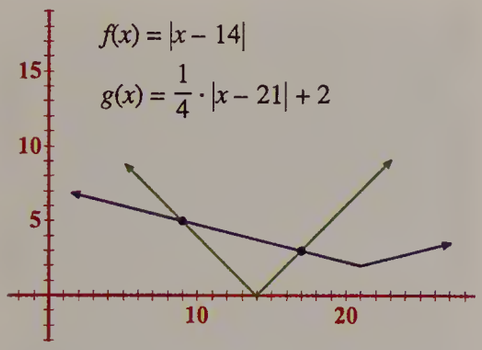


Alternate solution: Apply Euler's formula for plane graphs: regions = edges – vertices + 2. Since there are 6 cuts, there will be 12 vertices on the circle, each with degree 3. We have ${}_6C_2 = 15$ vertices inside the circle, each with degree 4. So we have a total of 27 vertices. The sum of the degrees is 96, and $96/2 = 48$ edges. Finally, $48 - 27 + 2 = 23$ regions, but this result includes the region outside the circular pizza, so we must subtract 1.

17. 19. The five integers consist of either three even and two odd integers, or two even and three odd integers. Suppose that there are two even integers. If a is the smaller even integer, then $a + a + 2 = 60 \rightarrow 2a = 58 \rightarrow a = 29$, a result that contradicts the premise that a is even. Therefore, we must have three even integers: $a + a + 2 + a + 4 = 60 \rightarrow 3a = 54 \rightarrow a = 18$. If 18, 20, and 22 are the even integers, 19 and 21 are the odd integers; 19 is the smaller odd integer.

18. 29. After squaring π and summing the four digits to the right of the decimal, we obtain $8 + 6 + 9 + 6 = 29$. The rational approximation of π^2, 9.869604401089…, is far less familiar than the decimal expansion of π. Some students may be surprised that the irrationality of π^2 had to be proved, but the product of two irrational numbers is not always irrational. Readers may wish to examine the following infinite series: $1/1^2 + 1/2^2 + 1/3^2 + 1/4^2 + \cdots$. In 1735 Euler proved that the exact sum is $\pi^2/6$.

19. 26. Let $f(x) = |x - 14|$, and let $g(x) = 0.25|x - 21| + 2$. Graph f and g in the coordinate plane and find the points of intersection: (9, 5) and (17, 3). The x-coordinates of these points are the solutions to the original equation; their sum is $9 + 17 = 26$. In addition, there is an algebraic solution that is somewhat more complicated. Consider three cases: $x \le 14$, $14 < x < 21$, and $x \ge 21$. In the third case, the equation becomes $x - 14 = 0.25(x - 21) + 2$. Solving for x produces a value of 43/3, contradicting the hypothesis that $x \ge 21$. If $x \le 14$, the equation becomes $14 - x = 0.25(21 - x) + 2$, producing the solution $x = 9$; and if $14 < x < 21$, the equation becomes $x - 14 = 0.25(21 - x) + 2$, producing $x = 17$.



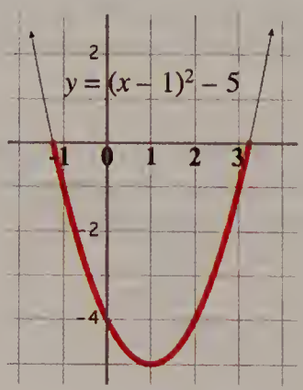
20. 25. One approach converts all the base 7 numbers to base 10 numbers. The question then becomes, How many positive integers between $2(343) + 7 + 2 = 695$ and $3(343) + 7 + 2 = 1038$ are divisible by $2(7) = 14$? The smallest multiple of 14 between 695 and 1038 is 700; the largest is 1036. Thus, $(1036 - 700)/14 + 1 = 25$. We add 1 because we need to count both 700 and 1036. Another approach works with the numbers in base 7: 2020_7 is the smallest multiple of 20_7 greater than 2012. From 2020 to 2200, we have 10_7 multiples of 20_7; from 2220_7 to 2400_7, we have another 10_7; and from 2420_7 to 2600_7; we have yet another 10_7. The multiples 2620_7, 2640_7, 2660_7, and 3010_7 bring us to the largest multiple of 20_7 less than 3012. The 34_7 multiples are equivalent to $3(7) + 4 = 25$ integers, as before.

Alternate solution: This approach also works with the number in base 7. Divide 2012_7 and 3012_7 by 20_7 to obtain 100_7 with remainder 12_7 and 134_7 with remainder 2_7, respectively. Subtracting $134_7 - 100_7 = 34_7 = 25_{10}$. Since both quotients had a remainder, we do not need to add 1.

21. 4. To obtain an even sum, we must have either three even addends or one even and two odd addends. The given set of integers contains five even integers and four odd integers. There are ${}_5C_3 = 10$ ways to choose three even integers. For each of the five even integers there are ${}_4C_2 = 6$ combinations of two different odd integers: $5 \cdot 6 = 30$. Thus, $A = 10 + 30 = 40$, and the sum of A's digits is 4.

22. 5. Changing the verbal statement into a mathematical open sentence, we have $m^2 - 2m < 4$. Complete the square: $m^2 - 2m + 1 < 5 \rightarrow (m - 1)^2 < 5$. Since m is an integer, $(m - 1) = 0, \pm 1, \pm 2$, implying that $m \in \{-1, 0, 1, 2, 3\}$. Thus, five integers satisfy the conditions.

Alternate solution: Completing the square again, we can write $(x - 1)^2 - 5 < 0$. Consider the related function $y = (x - 1)^2 - 5$. Graph the parabola with vertex $(1, -5)$ and observe where the graph intersects the x-axis. The y-values of the function are less than 0 between the intercepts. The figure shows the integers that lie between the intercepts.



23. 10. As n increases, Q increases faster than S. If the difference $Q - S > 2012$, then $Q > 2012$. Working backward, we note that the smallest square greater than 2012 is $2025 = 45^2$, and $45 = 1 + 2 + \cdots + 8 + 9$. We compare $Q_9 = 2025$ to $S_9 = 1^2 + 2^2 + \cdots + 8^2 + 9^2 = 285$: $2025 - 285 = 1740 < 2012$, so we need to look further. We find Q for $n = 10$. We add 45 and 10 to get 55, so $Q_{10} = 55^2 = 3025$, which we need to compare with $S_{10} = 285 + 10^2 = 385$. Subtracting, we have $Q_{10} - S_{10} = 3025 - 385 = 2640 > 2012$, so $n = 10$ provides our answer.

Alternate solution: The sum of the first n squares is $S_n = n^3/3 + n^2/2 + n/6$, and the square of the sum of the first n natural numbers is $Q_n = n^4/4 + n^3/2 + n^2/4$. We need $Q_n - S_n > 2012$, or $(n^4/4 + n^3/2 + n^2/4) - (n^3/3 + n^2/2 + n/6) > 2012 \rightarrow n^4/4 + n^3/6 - n^2/4 - n/6 > 2012 \rightarrow (n/12) \cdot (3n + 2)(n - 1)(n + 1) > 2012$. Compute values of the expression on the left side of the inequality to find the smallest n that satisfies the inequality: $n = 10$, as before.

24. 8. We recall that consecutive perfect squares differ by consecutive odd integers. If 251_b and 220_b differ by a, then 304_b and 251_b differ by $a + 2$. Create and solve a system of two equations:

$$2b^2 + 2b + a = 2b^2 + 5b + 1$$
$$2b^2 + 5b + 1 + a + 2 = 3b^2 + 4$$

Solving the first equation for a, we have $a = 3b + 1$; solving the second for a gives us $a = b^2 - 5b + 1$. Therefore, $b^2 - 5b + 1 = 3b + 1 \rightarrow b^2 - 8b = 0 \rightarrow b = 8$. (Reject $b = 0$.) We note that $220_8 = 144_{10}$, $251_8 = 169_{10}$, and $304_8 = 196$.

25. 15. It will be helpful to know that the total value of 1 penny, 1 nickel, 1 dime, and 1 quarter is 41 cents. There are 4 ways to select 3 of the same coin; those combinations have a total value of $3(41) = \$1.23$. There are 4 ways to select 3 different coins; each combination lacks one of the four coins. Therefore, the value of those combinations is also $3(41) = \$1.23$. There are 6 ways to choose 2 different coins: pennies and nickels, pennies and dimes, pennies and quarters, and so on. In each case, we

will have 2 of the first coin and 1 of the second coin or vice versa. Let p, n, d, and q represent the coins. We note that $(2p + 1n)$ and $(1p + 2n)$ total $3p + 3n$. If we listed all 12 combinations, we would have 9 of each coin, or a total of $9(41) = \$3.69$. Adding these partial results gives us $\$1.23 + \$1.23 + \$3.69 = \$6.15$, which exceeds \$6.00 by 15 cents.

Alternate solution: Count the number of times each of the different denominations appears in the combinations. Consider the penny, for example. There is 1 combination with 3 pennies, denoted *ppp*. There are 3 combinations with exactly 2 pennies, *pp**, where * denotes any coin other than a penny. There are 6 combinations with exactly 1 penny, *p***. Each coin will appear the same number of times in the 15 combinations: $15(\$0.41) = \$6.15$, which exceeds \$6.00 by 15 cents.

26. 28 ft.^{2}. The area of one base is 2.5 ft.^{2}. Since the volume is 10 ft.^{3} = $Bh = 2.5 \cdot h$, the height or lateral edge must be 4 ft. Let a represent the shorter base edge, and let $a + d$ represent the longer. Then $a(a + d) = 2.5$. We also have $a + 2d = 4$ since the dimensions are in arithmetic progression. The latter equation gives us $d = 2 - (a/2)$. Substitute as follows: $a(a + 2 - a/2) = 2.5 \rightarrow a^2/2 + 2a - 2.5 = 0 \rightarrow a^2 + 4a - 5 = 0 \rightarrow (a + 5) \cdot (a - 1) = 0 \rightarrow a = 1$ ft. The other base edge must be 2.5 ft. The lateral area is *perimeter* · *height* $= 2(2.5 + 1) \cdot 4 = 7 \cdot 4 = 28$ ft.^{2}.

Alternate solution: Since the height is 4 ft. and the three edges are in arithmetic progression, we let d be the common difference and represent the two edges of the base as $4 - d$ and $4 - 2d$. Then $(4 - d)(4 - 2d) = 2.5 \rightarrow d = 1.5$ (or 4.5, which is impossible), and the two missing edges are 2.5 and 1.

27. 11. If we examine the digits that precede the unknown a, we observe that 854 is divisible by 7. So the problem simplifies to finding values of a such that $4a5894$ is divisible by 7. We create the other factor or factors of $4a5894$ by completing the multiplication problem as shown:

$$\begin{array}{cccccc} & & 5 & 2 & 1 & \\ & & 0 & 8 & 4 & 2 \\ & & & & & 7 \\ \hline 4 & a & 5 & 8 & 9 & 4 \end{array}$$

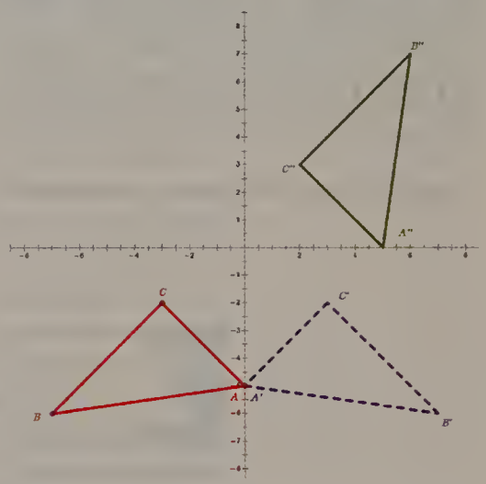
Since 5894 is also divisible by 7, the problem simplifies to finding multiples of 7 between 40 and 49. The two possibilities are 42 and 49, so $a = 2$ or $a = 9$; their sum is 11.

Alternate solution: Repeated use of one of the divisibility tests for 7 will also yield this answer. Probably the most common divisibility test for 7 is to excise the rightmost digit, double that digit, and subtract the product from the number that remains, repeating the process until a number appears that is small enough so that we can tell whether it is a multiple of 7. To check 854 in this way, double the 4 in the units place and subtract 8 from 85 to obtain 77, which is clearly divisible by 7. Following are the steps for checking $8544a5894$: $8544a5894 \rightarrow 8544a581 \rightarrow 8544a56$. Now we see that 56 is a multiple of 7, so we are concerned only with what is left: $8544a \rightarrow 854(4 - 2a) \rightarrow 85(4 - 2(4 - 2a)) = 85(4a - 4) = 840 + 10 + 4a - 4 = 840 + 6 + 4a$. Since 840 is a multiple of 7, $6 + 4a$ must also be a multiple of 7. Inspection leads us to 2 and 9 as the only values for a.

28. 6. The symbol -1 has three different meanings in the problem. When placed outside the brackets, the -1 symbol tells us to take the reciprocal of the expression inside the brackets. The symbols f^{-1} and g^{-1} indicate the inverses of the functions f and g. Within the parentheses, -1 is just a negative integer. To find the inverse of $f(x)$, we switch x and y: $x = 3y \rightarrow f^{-1}(x) = x/3$. To find the inverse of $g(x)$, we have $x = 1/y - 3 \rightarrow 1/y = x + 3 \rightarrow g^{-1}(x) = 1/(x + 3)$. The composition $(f^{-1} \circ g^{-1})(x) = (1/3) \cdot (1/(x + 3)) = 1/(3x + 9)$. At $x = -1$, we have $[1/(-3 + 9)]^{-1} = 6$.

29. 30. First, separate the integers into three different groups, also called residue classes, according to whether division by 3 leaves a remainder of 0, 1, or 2. The integers 12, 15, and 18 leave a remainder of 0; the integers 13, 16, and 19 leave a remainder of 1; the integers 14, 17, and 20 leave a remainder of 2. So each integer in the set has an associated remainder. If the sum of the three addends' remainders is divisible by 3, then the sum of the addends is divisible by 3. We can obtain a remainder sum of 0, 3, or 6 if we choose the three integers from the same group. We can also obtain a remainder sum of 3 if we choose one integer from each group. There are $3 \cdot 3 \cdot 3 = 27$ ways to choose one integer from each group. Thus, we have $1 + 1 + 1 + 27 = 30$ ways to obtain a multiple of 3.

30. 23. Reflection in the y-axis maps (x, y) to $(-x, y)$: $(0, -5) \rightarrow (0, -5)$; $(-7, -6) \rightarrow (7, -6)$; $(-3, -2) \rightarrow (3, -2)$. A 90° rotation about the origin maps (x, y) to $(-y, x)$: $(0, -5) \rightarrow (5, 0)$; $(7, -6) \rightarrow (6, 7)$; and $(3, -2) \rightarrow (2, 3)$. $\triangle A'B''C''$ has coordinates A' (5, 0), B'' (6, 7), and C'' (2, 3), and the sum of these coordinates is 23.



USING DISK AS MODELS FOR PROOFS

Students use balls and disks to prove the general formulas

Tongta Somchaipeng, Tussatrin Kruatong, and Bhinyo Panijpan

Exploring and deriving proofs of closed-form expressions for series can be fun for students. However, for some students, a physical representation of such problems is more meaningful.

Various approaches have been designed to help students visualize squares of sums and sums of squares; these approaches may be arithmetic-algebraic or combinatorial in nature, or they may involve mathematical induction. In his combinatorial approach, Stein (1971) counted the number of rectangles embedded in an $n \times n$ square ruled into n^2 smaller squares. One method yielded $\sum_{i=1}^{n} i^3$ rectangles, whereas the other yielded $\left(\sum_{i=1}^{n} i\right)^2$ rectangles. Mackiw (2000), also using a combinatorial approach, proved that $\sum_{i=1}^{n} i^p$ is a polynomial of degree $p + 1$ whose $(p - 4$ unknown) coefficients can be found by solving a system of $p - 4$ linear equations. Nelson (1993) assembled seven different visual proofs ("proofs without words") of sums of squares and sums of cubes. A model by Helfer and Won (1998), called "proof by tomatoes," is worth a look. Several other physical representations have been created that are conducive to proofs by induction (e.g., Shult and Tenenbaum 1988; Burroughs 2007).

Precalculus students have usually encountered a proof of the closed-form expression for the sum of the natural numbers from 1 to n, a sum that occurs in many situations. The double-count approach (Bezuszka and Kenney 2004–2005) is the most commonly used such proof:

$$2\sum_{i=1}^{n} i = 1+2+3+\cdots+(n-2)+(n-1)+n+$$
$$n+(n-1)+(n-2)+\cdots+3+2+1$$
$$= n(n+1)$$
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

We wanted our secondary school students to extend this knowledge to proofs of other series involving the summation of integers. Specifically, we explored the use of a physical model to prove by induction the relationship between the sum of the first n cubes and the square of the sum of the first n natural numbers—namely, to prove that

$$\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2.$$

problem to students, we asked them to think about how the blocks could be used in the induction proof anchor. At first, they were confused: How could 1^3 be made equal to 1^2 when the former represents a volume (a wooden cube) and the latter an area (only the face of the cube)?

To help students explore the relationship $1^3 + 2^3 + 3^3 + 4^3 + 5^3 = (1 + 2 + 3 + 4 + 5)^2$ and its generalization directly, we guided them to construct an alternative physical model that avoids the confusion caused by the different dimensions. We first reviewed with students sums such as $1 + 2 + 3 + 4$, $1 + 2 + 3 + 4 + \cdots + 15$, and $1 + 2 + 3 + 4 + \cdots + n$. Then we asked the following questions. (Activity sheets with these questions can be found at **www.nctm.org/mt021**.)

1. What is the total volume of balls with radii 1, 2, and 3 inches?
2. What is the total volume of balls with radii 1, 2, 3, ... , and 10 inches?
3. How would you calculate the total volume of 100 balls with radii 1, 2, 3, ... , and 100 inches?

Most students already knew how to calculate the volume of balls (spheres) and cylinders (disks), and this knowledge enabled them to investigate the relationship between $\sum_{i=1}^{n} i^3$ and $\left(\sum_{i=1}^{n} i\right)^2$. For this activity, we provided four different sizes of balls made of Plasticine® (a plastic modeling material). The blue, pink, yellow, and green balls had a radius of 1, 2, 3, and 4 units, respectively (see **fig. 2**).

First, students used the blue ball to construct a disk with a height of 4/3 units. Thus, the top surface area of the disk is $\pi \cdot 1^2$ (see **fig. 3a**). Second, students expanded the radius of the disk by flattening the pink ball to form a ring with inner radius 1, outer radius 3, and height 4/3 units (see **fig. 3b**). The volume difference between the new disk (blue and pink) and the inner one (blue) is $4\pi(3^2)/3 - 4\pi(1^2)/3 = 4\pi(2^3)/3$—that is, the initial volume of the pink ball. The surface area of the disk made from the two balls becomes 9π.

Fig. 2 These four balls, with a radius of 1, 2, 3, and 4 units, respectively, have a total volume of $(4\pi/3) \cdot (1^3 + 2^3 + 3^3 + 4^3)$ units^3.

OF SERIES

ıms of squares and cubes.

USING A CIRCULAR DISK

Biesecker (Burroughs 2007) devised a visual representation (resembling that of Fry [1993]) to show the connection between the sum of cubes and the square of a sum (see **fig. 1**). When introducing this

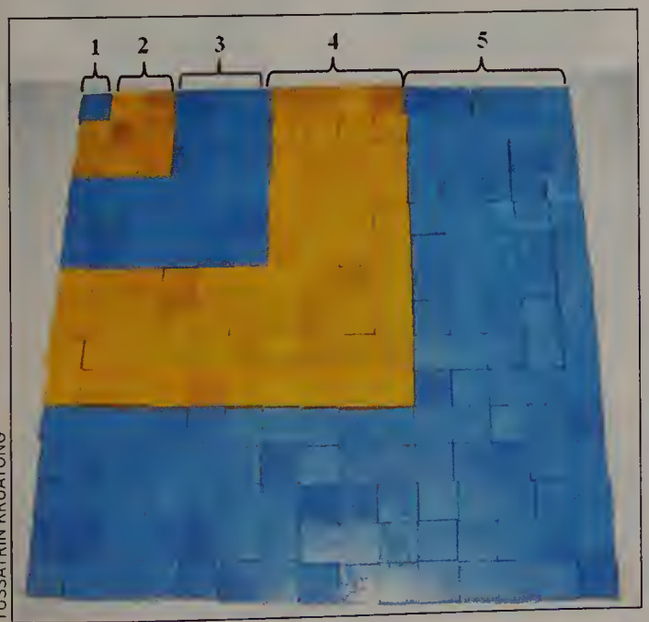


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Fig. 1 A set of square wooden slabs shows that $1^3 + 2^3 + 3^3 + 4^3 + 5^3 = (1 + 2 + 3 + 4 + 5)^2$.

Next, students flattened the yellow ball into another ring with inner radius 3 and outer radius 6 units and added it to the existing disk. We calculated the volume of the new yellow ring as $4\pi(6^2)/3 - 4\pi(3^2)/3 = 4\pi(3^3)/3$—that is, the initial volume of the yellow ball. The surface area of the disk formed from the three colored balls now becomes $\pi(6^2) = \pi(1 + 2 + 3)^2$. We then asked students to predict the radius of the next disk if the green ball were added to the existing disk (see **fig. 4**).

Through this process, students learned how to show the relationship between the radius of the Plasticine balls and the radius of the disks (see **table 1**). Students concluded that the sum of the cubed radial values of the spheres is equal to the square of the combined radius of the disk made from these spheres with a height of 4/3 units. More explicitly, they obtained the relationship $1^3 + 2^3 + 3^3 + 4^3 = (1 + 2 + 3 + 4)^2$. Students were guided to continue exploring this particular visual representation until they arrived at the following formula:

$$\frac{4}{3}\pi\left(1^3\right)+\frac{4}{3}\pi\left(2^3\right)+\frac{4}{3}\pi\left(3^3\right)+\cdots+\frac{4}{3}\pi\left(n^3\right)=\pi\cdot\left(1+2+3+\cdots+n\right)^2\cdot\frac{4}{3}$$

$$\left(1^3+2^3+3^3+\cdots+n^3\right)=\left(1+2+3+\cdots+n\right)^2$$

$$\sum n^3=\left(\sum n\right)^2=\left[\frac{n\left(n+1\right)}{2}\right]^2$$

This process of de-forming balls into disks does not involve taking apart or putting together component pieces. Students were confident throughout that the volume did not change.

We then proposed a problem along the lines of turning swords into plowshares: Turn solid cannonballs with radius of 3, 4, 5, 6, 7, 8, and 9 inches, respectively, into a circular tablet for the school's honor roll. If the tablet has a thickness of 4/3 inches, what would be its radius?

THE SUM OF SQUARES INVOLVING DISKS AND CONES

Many variations of this theme of spheres and disks are possible. For example, students can make a stepped cone by stacking disks of unit height and with radius of 1 to n units, respectively, from the top disk to the bottom disk. A right circular cone can then be made from this stepped cone by stripping away excess Plasticine. This disk-to-cone procedure can serve as compelling physical evidence for the following formula:

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n\left(n+1\right)\left(2n+1\right)}{6}$$

For this activity, students are given disks of different sizes and radii that differ incrementally by 1 unit. In **figure 5**, the yellow, pink, blue, orange, and green disks have a radius of 1, 2, 3, 4, and 5 units, respectively, and are stacked one atop the next.

The general formula for a cone with a base of radius r and height h is $\pi r^2 h/3$. In other words, a cone has the volume of one-third of a cylinder of the same base radius and height. To make a cone from stacked disks, the edge of each disk must be cut off along lines from the center of the top part of the yellow disk to the base perimeter of the green disk (see **fig. 6**). The yellow disk is thus the apex of the smooth cone.

The stepped cones encompass the smooth cones.

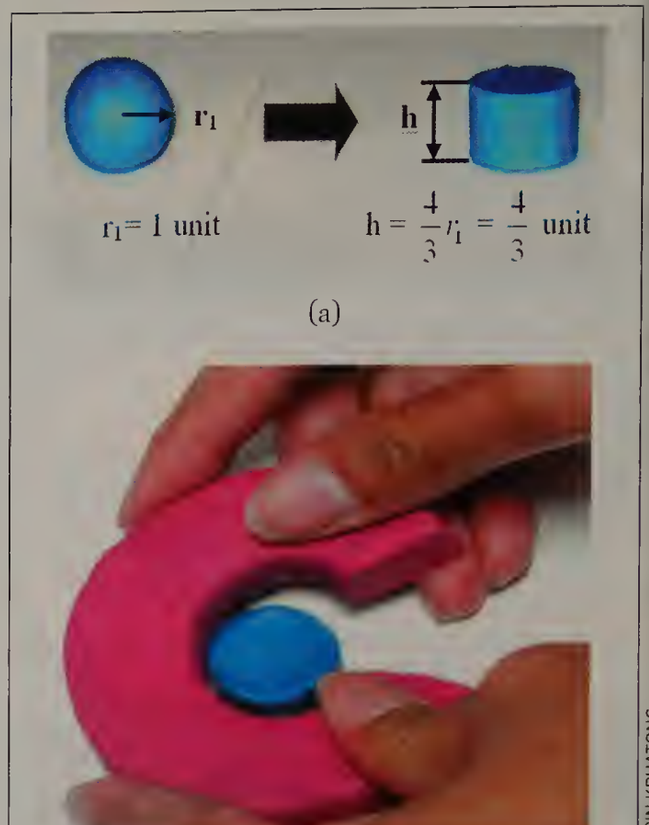

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Fig. 3 From the blue ball, students made a disk with a height of 4/3 units; the ball and the disk will have the same volume (**a**). The expanded radius of a new disk can be made by fitting a ring made from the flattened pink ball ($r = 2$) around the existing disk (**b**).

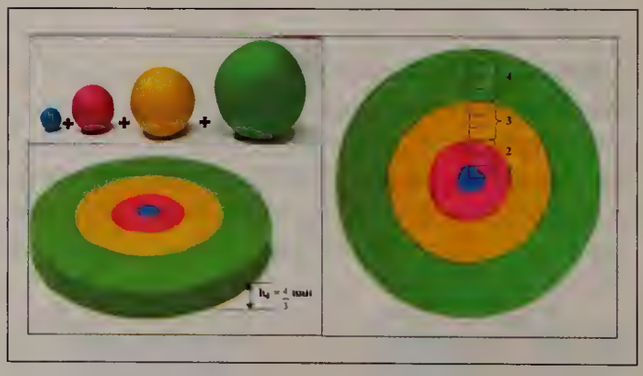

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Fig. 4 Adding the green ring makes a composite disk with radius 1 + 2 + 3 + 4 units.

Table 1 Volumes of Balls and Disks

Step	Volume of Ball	Radius of Disk	Volume of Disk
1	$\frac{4}{3}\pi\left(1^3\right)$	1	$\pi\left(1^2\right)\left(\frac{4}{3}\right)$
2	$\frac{4}{3}\pi\left(1^3\right)+\frac{4}{3}\pi\left(2^3\right)$	3 *or* 1 + 2	$\pi\left(3^2\right)\left(\frac{4}{3}\right)$
3	$\frac{4}{3}\pi\left(1^3\right)+\frac{4}{3}\pi\left(2^3\right)+\frac{4}{3}\pi\left(3^3\right)$	6 *or* 1 + 2 + 3	$\pi\left(6^2\right)\left(\frac{4}{3}\right)$
4	$\frac{4}{3}\pi\left(1^3\right)+\frac{4}{3}\pi\left(2^3\right)+\frac{4}{3}\pi\left(3^3\right)+\frac{4}{3}\pi\left(4^3\right)$	10 *or* 1 + 2 + 3 + 4	$\pi\left(10^2\right)\left(\frac{4}{3}\right)$
⋮	⋮	⋮	⋮
n	$\frac{4}{3}\pi\left(1^3\right)+\frac{4}{3}\pi\left(2^3\right)+\frac{4}{3}\pi\left(3^3\right)+\cdots+\frac{4}{3}\pi\left(n^3\right)$	$1+2+3+\cdots+n$	$\pi\cdot(1+2+3+\cdots+n)^2\cdot\frac{4}{3}$

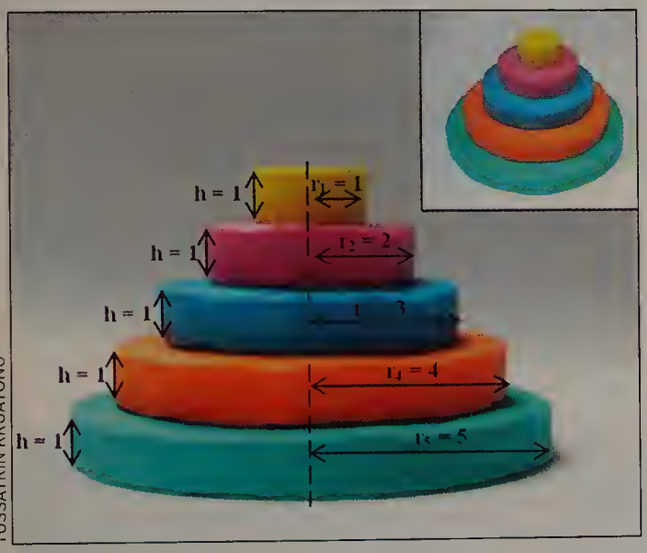

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Fig. 5 The total volume of the cone is $\pi(1^2)(1) + \pi(2^2)(1) + \pi(3^2)(1) + \pi(4^2)(1) + \pi(5^2)(1)$.

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Fig. 6 A cone can be made from a stepped cone by drawing lines joining the apex with the perimeter of the base and removing excess material.

Consider the result of subtracting each cone's volume from the volume of the stepped cone that surrounds it. The first subtraction, $\pi \cdot 1^2 - \pi \cdot 1^3/3$, yields $2\pi/3$; the second subtraction yields $7\pi/3$, or $2\pi/3 + 5\pi/3 = (\pi \cdot 1^2 + \pi \cdot 2^2) - \pi \cdot 2^3/3$; and so on. **Table 2** (p. 50) lists the volumes and their differences; the results make the pattern more obvious. The total volume of the trimmed part can be written in a general form as the sum of $\pi(3n - 1)/3$. From this particular visual representation, students can obtain the sum of squares as the final equation:

$$\left[\pi\left(1^2\right)+\pi\left(2^2\right)+\pi\left(3^2\right)+\cdots+\pi\left(n^2\right)\right]-\frac{1}{3}\pi\left(n^3\right)=\frac{2}{3}\pi+\frac{5}{3}\pi+\frac{8}{3}\pi+\cdots+\frac{3n-1}{3}\pi$$

$$1^2+2^2+3^2+\cdots+n^2-\frac{1}{3}n^3=\frac{2}{3}+\frac{5}{3}+\frac{8}{3}+\cdots+\frac{3n-1}{3}$$

$$\sum n^2-\frac{n^3}{3}=\sum\frac{3n-1}{3}$$

$$\sum n^2=\frac{n^3}{3}+\sum\frac{3n-1}{3}=\frac{n^3}{3}+\sum n-\frac{n}{3}=\frac{n^3}{3}+\frac{n\left(n+1\right)}{2}-\frac{n}{3}$$

$$=\frac{2n^3+3n^2+3n-2n}{6}=\frac{n\left(n+1\right)\left(2n+1\right)}{6}$$

The following problem might be used to check whether students can apply what they have learned:

> A mathematician couple were planning their wedding cake. Both wanted a nine-tiered cake with the top tier having a radius of 3 inches and successive tiers having a radial difference of one inch. The bottom tier would then have an 11-inch radius. The groom, however, wanted the height of each tier to equal the radius, whereas the bride wanted the height of each tier to be 7 inches. Which tiered cake would be heavier?

Table 2 Volumes of Cones from Stacked Disks

Step	Volume of Stacked Disks (V_D) ($h = 1$ unit)	Volume of Idealized Cone (V_C)	$V_D - V_C$
1	$\pi\left(1^2\right)$	$\frac{1}{3}\pi\left(1^3\right)$	$\frac{2}{3}\pi$
2	$\pi\left(1^2\right)+\pi\left(2^2\right)$	$\frac{1}{3}\pi\left(2^3\right)$	$\frac{7}{3}\pi$ or $\frac{2}{3}\pi+\frac{5}{3}\pi$
3	$\pi\left(1^2\right)+\pi\left(2^2\right)+\pi\left(3^2\right)$	$\frac{1}{3}\pi\left(3^3\right)$	$\frac{15}{3}\pi$ or $\frac{2}{3}\pi+\frac{5}{3}\pi+\frac{8}{3}\pi$
4	$\pi\left(1^2\right)+\pi\left(2^2\right)+\pi\left(3^2\right)+\pi\left(4^2\right)$	$\frac{1}{3}\pi\left(4^3\right)$	$\frac{26}{3}\pi$ or $\frac{2}{3}\pi+\frac{5}{3}\pi+\frac{8}{3}\pi+\frac{11}{3}\pi$
5	$\pi\left(1^2\right)+\pi\left(2^2\right)+\pi\left(3^2\right)+\pi\left(4^2\right)+\pi\left(5^2\right)$	$\frac{1}{3}\pi\left(5^3\right)$	$\frac{40}{3}\pi$ or $\frac{2}{3}\pi+\frac{5}{3}\pi+\frac{8}{3}\pi+\frac{11}{3}\pi+\frac{14}{3}\pi$
⋮	⋮	⋮	⋮
n	$\pi\left(1^2\right)+\pi\left(2^2\right)+\pi\left(3^2\right)+\cdots+\pi\left(n^2\right)$	$\frac{1}{3}\pi\left(n^3\right)$	$\frac{2}{3}\pi+\frac{5}{3}\pi+\frac{8}{3}\pi+\cdots+\frac{3n-1}{3}\pi$

CONCLUSION

By varying the kinds of proofs we use, we can take into consideration differences in students' learning styles. This new approach to proving algebraic and number-theoretical relationships using geometric volumes also lends itself to inductive proof. As a result, with this approach students are able to see more and do more on their own.

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Editor's note: For another take on sums of powers, see Delving Deeper in this issue (pp. 70–77).

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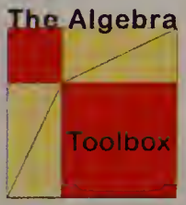

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Joel A. Bryan

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During the thirteen years that I taught high school physics and mathematics, I found that my physics students typically came to class excited to learn. As in all science classes, they interacted with fellow classmates while performing laboratory investigations and other group activities requiring higher-order thinking skills. To create a similar experience for my mathematics students, I developed a laboratory investigation for my precalculus class. These students responded just as favorably as my physics students to hands-on data collection activities.

The following investigation illustrates one real-world application of exponential functions and graphs. In this activity, which is appropriate for algebra, precalculus, and calculus students studying exponential functions, students use random numbers to simulate the breakdown of unstable atomic nuclei—radioactive decay. They use the data to determine a best-fit equation and compare that equation with theoretical data.

THE ACTIVITY

Collecting the Data

The data collection depends on random numbers, and graphing calculators and computer software can generate pseudo-random numbers. My students use a random digits chart (see **fig. 1**) to collect experimental data. They then compare these data with data computed using the theoretical decay probability, which we have conveniently set at 10 percent.

To begin this process, students select a block of digits on the chart in **figure 1** to represent a specified number of radioactive nuclei. We have selected the digit 1 to represent a decaying nucleus from an initial sample of 500 unstable nuclei (random digits), although other choices can be made for each. The experimental number of nuclei that decay during any time period will be equal to the number of times that 1 appears in the selected block of digits. Each stage requires a new block of digits to represent nuclei that remain undecayed, but any block of digits will suffice.

The process is illustrated in **figure 2**. For convenience, I begin several blocks using a new row of digits. In these cases, I have drawn a line through unused digits in the previous row to avoid confusion with the active data at that stage.

In the example shown in **figure 2**, the first 500 digits in the chart are initially selected. This block has 38 1s, meaning that 38 nuclei decayed in the initial time period. The second block begins in row 00010 and contains the remaining 462 digits. This block has 47 1s, leaving 415 undecayed nuclei. So the first three data points will be (0, 500), (1, 462), and (2, 415). This process continues until a satisfactory number of data points have been collected.

Graphing the Data

After collecting the data, students use graphing calculators or computer spreadsheets to graph both the theoretical and the experimental data (see **figs. 3** [p. 55] and **4** [p. 56]). As students will see, their experimental data may differ significantly from the theoretical data. In addition to comparing theoretical and experimental data, students can use their data tables, graphs, and equations to determine half-lives from both the graphical and mathematical models.

This activity requires no special equipment. All that is needed to model radioactive nuclei is a table of random digits, which may be obtained from statistics textbooks or from numerous on-line sources. We could also generate lists of pseudorandom digits in Microsoft Excel® using the command `=INT(10*RAND())`, which generates the digits 0–9 and gives each digit a decay

Random Digits Chart

http://www.rand.org/publications/classics/randomdigits/randomdata.html

00000	10097	32533	76520	13586	34673	54876	80959	09117	39292	74945
00001	37542	04805	64894	74296	24805	24037	20636	10402	00822	91665
00002	08422	68953	19645	09303	23209	02560	15953	34764	35080	33606
00003	99019	02529	09376	70715	38311	31165	88676	74397	04436	27659
00004	12807	99970	80157	36147	64032	36653	98951	16877	12171	76833
00005	66065	74717	34072	76850	36697	36170	65813	39885	11199	29170
00006	31060	10805	45571	82406	35303	42614	86799	07439	23403	09732
00007	85269	77602	02051	65692	68665	74818	73053	85247	18623	88579
00008	63573	32135	05325	47048	90553	57548	28468	28709	83491	25624
00009	73796	45753	03529	64778	35808	34282	60935	20344	35273	88435
00010	98520	17767	14905	68607	22109	40558	60970	93433	50500	73998
00011	11805	05431	39808	27732	50725	68248	29405	24201	52775	67851
00012	83452	99634	06288	98083	13746	70078	18475	40610	68711	77817
00013	88685	40200	86507	58401	36766	67951	90364	76493	29609	11062
00014	99594	67348	87517	64969	91826	08928	93785	61368	23478	34113
00015	65481	17674	17468	50950	58047	76974	73039	57186	40218	16544
00016	80124	35635	17727	08015	45318	22374	21115	78253	14385	53763
00017	74350	99817	77402	77214	43236	00210	45521	64237	96286	02655
00018	69916	26803	66252	29148	36936	87203	76621	13990	94400	56418
00019	09893	20505	14225	68514	46427	56788	96297	78822	54382	14598
00020	91499	14523	68479	27686	46162	83554	94750	89923	37089	20048
00021	80336	94598	26940	36858	70297	34135	53140	33340	42050	82341
00022	44104	81949	85157	47954	32979	26575	57600	40881	22222	06413
00023	12550	73742	11100	02040	12860	74697	96644	89439	28707	25815
00024	63606	49329	16505	34484	40219	52563	43651	77082	07207	31790
00025	61196	90446	26457	47774	51924	33729	65394	59593	42582	60527
00026	15474	45266	95270	79953	59367	83848	82396	10118	33211	59466
00027	94557	28573	67897	54387	54622	44431	91190	42592	92927	45973
00028	42481	16213	97344	08721	16868	48767	03071	12059	25701	46670
00029	23523	78317	73208	89837	68935	91416	26252	29663	05522	82562
00030	04493	52494	75246	33824	45862	51025	61962	79335	65337	12472
00031	00549	97654	64051	88159	96119	63896	54692	82391	23287	29529
00032	35963	15307	26898	09354	33351	35462	77974	50024	90103	39333
00033	59808	08391	45427	26842	83609	49700	13021	24892	78565	20106
00034	46058	85236	01390	92286	77281	44077	93910	83647	70617	42941
00035	32179	00597	87379	25241	05567	07007	86743	17157	85394	11838
00036	69234	61406	20117	45204	15956	60000	18743	92423	97118	96338
00037	19565	41430	01758	75379	40419	21585	66674	36806	84962	85207
00038	45155	14938	19476	07246	43667	94543	59047	90033	20826	69541
00039	94864	31994	36168	10851	34888	81553	01540	35456	05014	51176
00040	98086	24826	45240	28404	44999	08896	39094	73407	35441	31880
00041	33185	16232	41941	50949	89435	48581	88695	41994	37548	73043
00042	80951	00406	96382	70774	20151	23387	25016	25298	94624	61171
00043	79752	49140	71961	28296	69861	02591	74852	20539	00387	59579
00044	18633	32537	98145	06571	31010	24674	05455	61427	77938	91936
00045	74029	43902	77557	32270	97790	17119	52527	58021	80814	51748
00046	54178	45611	80993	37143	05335	12969	56127	19255	36040	90324
00047	11664	49883	52079	84827	59381	71539	09973	33440	88461	23356
00048	48324	77928	31249	64710	02295	36870	32307	57546	15020	09994
00049	69074	94138	87637	91976	35584	04401	10518	21615	01848	76938
00050	09188	20097	32825	39527	04220	86304	83389	87374	64278	58044
00051	90045	85497	51981	50654	94938	81997	91870	76150	68476	64659
00052	73189	50207	47677	26269	62290	64464	27124	67018	41361	82760
00053	75768	76490	20971	87749	90429	12272	95375	05871	93823	43178
00054	54016	44056	66281	31003	00682	27398	20714	53295	07706	17813
00055	08358	69910	78542	42785	13661	58873	04618	97553	31223	08420
00056	28306	03264	81333	10591	40510	07893	32604	60475	94119	01840
00057	53840	86233	81594	13628	51215	90290	28466	68795	77762	20791
00058	91757	53741	61613	62269	50263	90212	55781	76514	83483	47055
00059	89415	92694	00397	58391	12607	17646	48949	72306	94541	37408
00060	77513	03820	86864	29901	68414	82774	51908	13980	72893	55507
00061	19502	37174	69979	20288	55210	29773	74287	75251	65344	67415
00062	21818	59313	93278	81757	05686	73156	07082	85046	31853	38452
00063	51474	66499	68107	23621	94049	91345	42836	09191	08007	45449
00064	99559	68331	62535	24170	69777	12830	74819	78142	43860	72834
00065	33713	48007	93584	72869	51926	64721	58303	29822	93174	93972
00066	85274	86893	11303	22970	28834	34137	73515	90400	71148	43643
00067	84133	89640	44035	52166	73852	70091	61222	60561	62327	18423
00068	56732	16234	17395	96131	10123	91622	85496	57560	81604	18880
00069	65138	56806	87648	85261	34313	65861	45875	21069	85644	47277
00070	38001	02176	81719	11711	71602	92937	74219	64049	65584	49698
00071	37402	96397	01304	77586	56271	10086	47324	62605	40030	37438
00072	97125	40348	87083	31417	21815	39250	75237	62047	15501	29578
00073	21826	41134	47143	34072	64638	85902	49139	06441	03856	54552
00074	73135	42742	95719	09035	85794	74296	08789	88156	64691	19202
00075	07638	77929	03061	18072	96207	44156	23821	99538	04713	66994
00076	60528	83441	07954	19814	59175	20695	05533	52139	61212	06455
00077	83596	35655	06958	92983	05128	09719	77433	53783	92301	50498
00078	10850	62746	99599	10507	13499	06319	53075	71839	06410	19362
00079	39820	98952	43622	63147	64421	80814	43800	09351	31024	73167
00080	59580	06478	75569	78800	88835	54486	23768	06156	04111	08408
00081	38508	07341	23793	48763	90822	97022	17719	04207	95954	49953
00082	30692	70668	94688	16127	56196	80091	82067	63400	05462	69200
00083	65443	95659	18288	27437	49632	24041	08337	65676	96299	90836
00084	27267	50264	13192	72294	07477	44606	17985	48911	97341	30358
00085	91307	06991	19072	24210	36699	53728	28825	35793	28976	66252
00086	68434	94688	84473	13622	62126	98408	12843	82590	09815	93146
00087	48908	15877	54745	24591	35700	04754	83824	52692	54130	55160
00088	06913	45197	42672	78601	11883	09528	63011	98901	14974	40344
00089	10455	16019	14210	33712	91342	37821	88325	80851	43667	70883
00090	12883	97343	65027	61184	04285	01392	17974	15077	90712	26769
00091	21778	30976	38807	36961	31649	42096	63281	02023	08816	47449
00092	19523	59515	65122	59659	86283	68258	69572	13798	16435	91529
00093	67245	52670	35583	16563	79246	86686	76463	34222	26655	90802
00094	60584	47377	07500	37992	45134	26529	26760	83637	41326	44344
00095	53853	41377	36066	94850	58838	73859	49364	73331	96240	43642
00096	24637	38736	74384	89342	52623	07992	12369	18601	03742	83873
00097	83080	12451	38992	22815	07759	51777	97377	27585	51972	37867
00098	16444	24334	36151	99073	27493	70939	85130	32552	54846	54759
00099	60790	18157	57178	65762	11161	78576	45819	52979	65130	04860
00100	03991	10461	93716	16894	66083	24653	84609	58232	88618	19161

Fig. 1 Students use the random digits chart to collect experimental data.

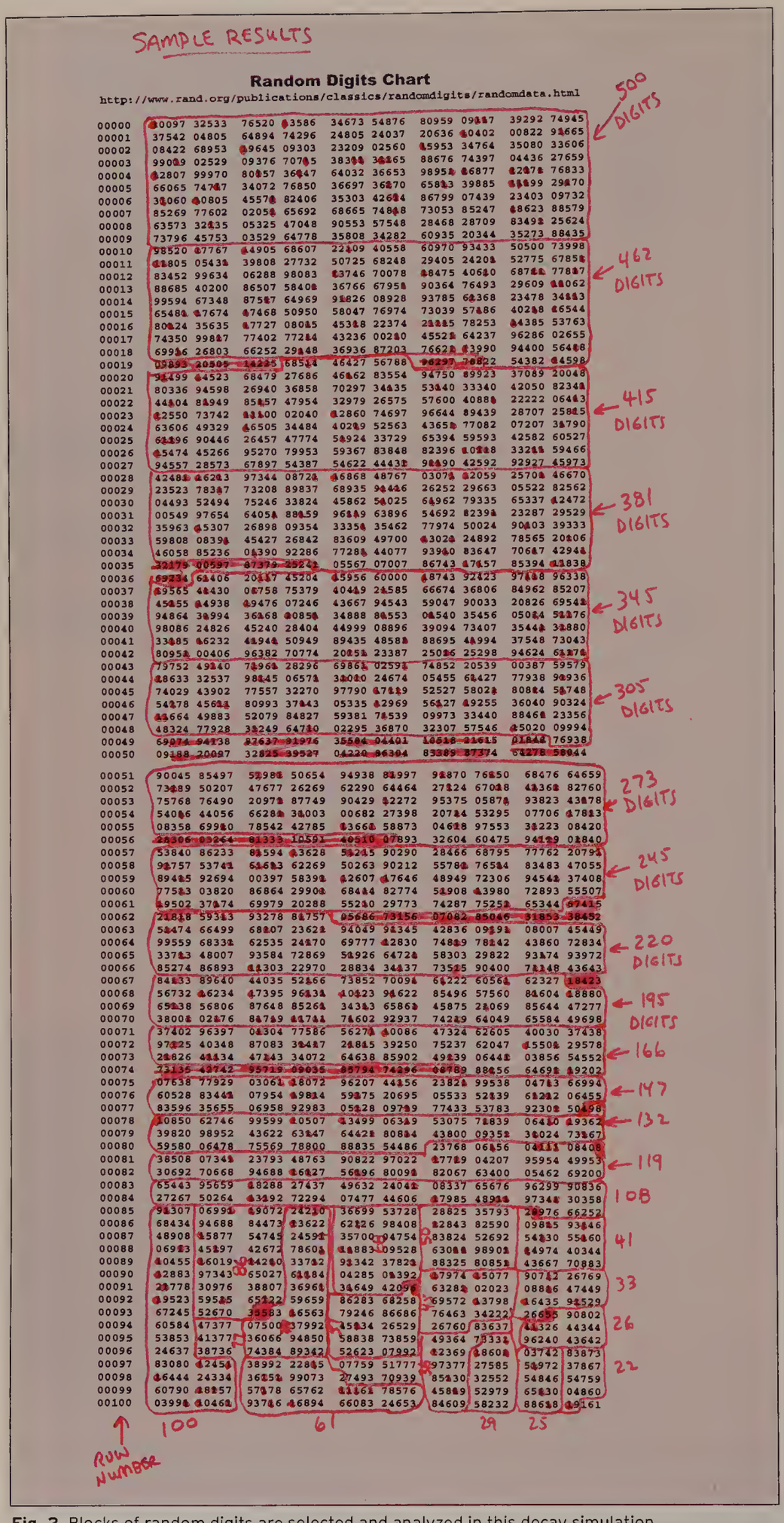

Fig. 2 Blocks of random digits are selected and analyzed in this decay simulation.

probability of 1/10. The command **RANDBETWEEN(0,9)** can also be used.

An advantage of using Excel is that changing the decay probability is easy: **RANDBETWEEN(0,8)** can be used if the decay probability is 1/9, **RANDBETWEEN(0,7)** if the decay probability is 1/8, **RANDBETWEEN(0,15)** if the decay probability is 1/16, and so on. Performing this exercise with varying theoretical decay rates would allow students to see how changes in the theoretical decay probability changes the theoretical and experimental data, graphs, equations, and half-lives.

ADDITIONAL ANALYSES

In addition to generating data that result in exponential plots and equations, this activity can be the basis for at least two additional interesting exercises for students. The tasks require students first to change the base of the exponential function and then to graph the data on semilog graph paper and obtain the exponential model from the graph (**fig. 5** [p. 56] provides an example).

Changing the Base of the Exponential Equation

Excel uses base e for the exponential equations it generates rather than a base of (1 – theoretical decay rate). Converting the theoretical equation to base e instead of 0.90 is a good exercise for students (and a skill expected of precalculus students). If we use the model from this activity, the equation $N(t) = 500(0.90)^t$ becomes $N(t) = 500e^{-0.1054t}$. This equation closely parallels the theoretical equation generated by Excel from the collected data; the slight differences may be attributed to the fact that the theoretical decay values in the data table were rounded to the nearest integer.

Plotting Data on Semilog Graph Paper

Another useful exercise is to plot the experimental and theoretical data on semilog graph paper (which has a logarithmic y-axis and a linear x-axis), draw a best-fit line and determine its slope and y-intercept, and then calculate the decay equation. Students could also use a spreadsheet such as Excel to generate the graph and the equation.

Students may be surprised to find that on this different type of graph paper the original data now yield a line with a positive y-intercept and a negative slope. Relevant discussion questions to incorporate at this point include but are not limited to these:

- What does the y-intercept of this graph represent?
- Is it possible for the y-intercept to be less than or equal to zero?
- Why is the slope negative?
- Is it possible for the slope to be positive?
- What would it mean if the graph had a positive slope?
- How would the graph change if the decay rate increased? If it decreased?
- If you were given both types of graphs and asked to determine the sample's half-life, which type of graph would you find easier to use?
- Can you name other real-world phenomena that, when graphed on semilog paper, would result in linear graphs?

If the data points are linear on semilog graph paper, the equation of the line will be in the form of $Y = Mx + B$ where $Y = \ln y$. Students can work back to the exponential equation by finding the slope of the line, $M \approx -0.1055$, and then the y-intercept, $B = \ln 500$. Then they can use each side of the linear equation as an exponent on e:

$$\ln y = -0.1055x + \ln 500$$
$$e^{\ln y} = e^{-0.1055x + \ln 500}$$
$$e^{\ln y} = e^{-0.1055x} \cdot e^{\ln 500}$$
$$y = 500e^{-0.1055x}$$

This result is remarkably close to the theoretical equation generated by Excel.

CONCLUSION

My class spent three periods collecting and analyzing radioactive decay data. Not only did students practice working with exponential functions, they also gained firsthand knowledge of the relevance of the mathematical skills involved. In addition, they enjoyed these classes.

Many other data collection activities that require no special materials can be used to introduce or reinforce mathematical concepts. Sources of ideas are

	Random Numbers: Theoretical Decay			Random Numbers: Experimental Decay		
Elapsed Time	Initial Number Present	Number Decayed	Number Remaining	Initial Number Present	Number Decayed	Number Remaining
0	500	50	450	500	38	462
1	450	45	405	462	47	415
2	405	41	364	415	34	381
3	364	36	328	381	36	345
4	328	33	295	345	40	305
5	295	30	265	305	32	273
6	265	27	238	273	28	245
7	238	24	214	245	25	220
8	214	21	193	220	25	195
9	193	19	175	195	29	166
10	175	17	157	166	19	147
11	157	16	141	147	15	132
12	141	14	127	132	13	119
13	127	13	114	119	11	108
14	114	11	103	108	8	100
15	103	10	93	100	14	86
16	93	9	85	86	9	77
17	85	8	76	77	9	68
18	76	8	68	68	7	61
19	68	7	61	61	10	51
20	61	6	55	51	1	50
21	55	6	49	50	5	45
22	49	5	44	45	4	41
23	44	4	40	41	5	36
24	40	4	36	36	3	33
25	36	4	32	33	4	29
26	32	3	29	29	3	26
27	29	3	26	26	1	25
28	26	3	23	25	3	22
29	23	2	21	22	1	21
30	21	2	19	21	0	21
31	19	2	17	21	2	19
32	17	2	15	19	2	17

Fig. 3 This data table contains the experimental decay results from **figure 2**. A blank table is available at **www.nctm.org/mt024**.

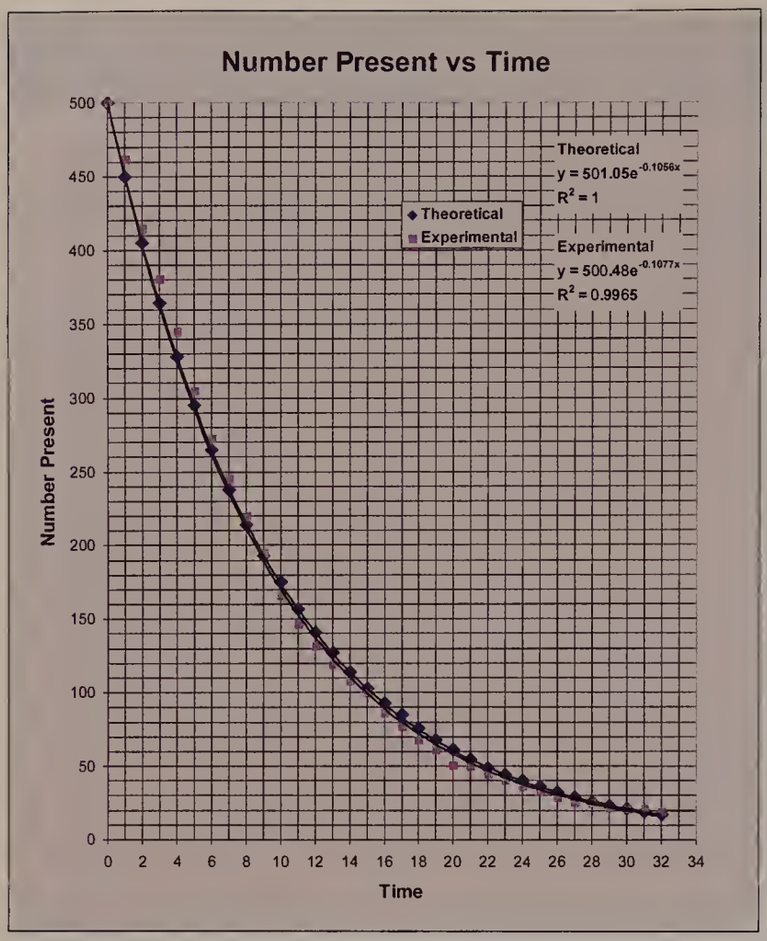

Fig. 4 The graphs of experimental and theoretical decay are very similar.

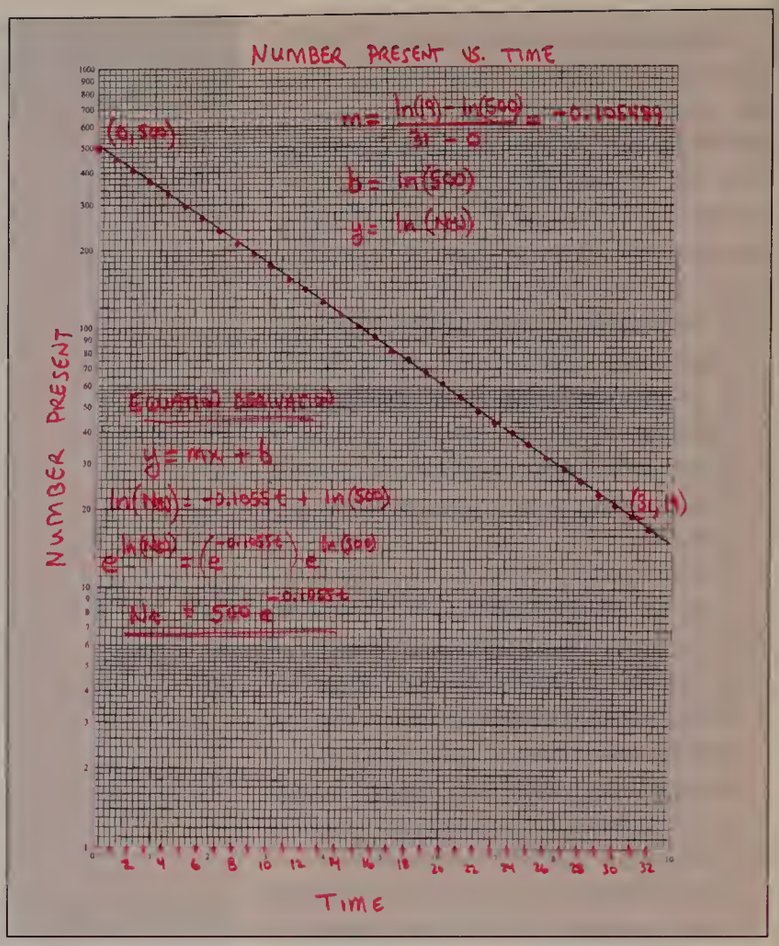

Fig. 5 Working backward from the graph on semilog graph paper, students can recover the exponential equation.

numerous, including the school's physics teacher, who may have special equipment available for borrowing.

As I lead workshops and make presentations, I tell this story often to encourage mathematics teachers to implement similar activities in their classes. Students will benefit from collecting and analyzing data, and mathematics class will become more enjoyable for them all.

SAMPLE RESPONSES TO ACTIVITY DISCUSSION QUESTIONS

1. Yes. Graphs coincide closely but not exactly.

2. The length of time it takes 100 unstable nuclei to decay depends on how many were initially present. It takes approximately two time periods for the number of nuclei to decay from 500 to 400 and four time periods for the number of nuclei to decay from 300 to 200; however, in seventeen time periods, the sample never decayed from 100 to 0.

3. 6.5; 6.5.

4. It appears to take approximately 6.5 time periods for the number of nuclei to decay from 400 to 250, from 300 to 150, and from 200 to 100.

5. 6.5 time periods.

6. We expect our chosen number to represent 10 percent of any group of random digits; therefore, 90 percent will remain, and the number present at any time t should be (0.90)(number present at time $t - 1$). Therefore, the theoretical decay equation for the number present at any time t will be $N(t) = N_0(0.90)^t$ where N_0 = number initially present. Our theoretical equation is therefore $N(t) = 500(0.90)^t$.

   The half-life is the length of time needed for $N(t) = 0.5N_0$. Therefore, $0.5N_0 = N_0\,(0.90)^t$. This equation reduces to $0.5 = 0.90^t$. Taking the logarithm of both sides, we have $\log(0.5) = \log(0.90^t) = t\log(0.90)$. Therefore, $t = \log(0.5)/\log(0.90) = -0.301030/-0.045757 = 6.58$. This calculation is relatively close to the value obtained by approximating values on the graph.

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For downloadable, customizable student activity sheets, go to the *Mathematics Teacher* website: **www.nctm.org/mt.**

For a blank table of random numbers, download one of the free apps for your smartphone and then scan this tag to access. **www.nctm.org/mt024.**

Activity 1: Using Random Numbers to Simulate Radioactive Decay

Radioactivity can be defined as the spontaneous uncontrollable decay of an atomic nucleus with the emission of particles or rays. The rate at which these unstable nuclei break down depends on the number present at any given time. Therefore, we expect that a certain percentage of the nuclei present will decay during each time period, but there is no way to determine which particular nuclei will decay.

You will use a table of random numbers to represent radioactive nuclei and investigate the rate at which they decay. If 1000 digits from 0 to 9 are randomly generated, theoretically we would expect that there would be 100 of each digit. In reality, however, there would likely *not* be 100 of each digit. Similarly, we might expect 10 percent of radioactive nuclei to decay during any particular time period, but in reality not exactly 10 percent would be decayed.

Instructions

1. Complete the theoretical decay entries in your table. Assume that exactly 1/10 of all the undecayed number present decay in each time period as you calculate the number decayed and the undecayed number remaining. Round answers to the nearest unit as necessary. Continue with 10 percent decaying until the table is complete or until fewer than 10 nuclei remain undecayed.

2. Make a graph of the number of theoretical radioactive nuclei present vs. time. Plot the number of radioactive nuclei present on the y-axis and time on the x-axis. Draw a smooth curve through the data points.

3. Now repeat this process using random digits to represent the unstable nuclei. Begin with 500 random digits. These represent 500 unstable nuclei, each equally likely to decay during the next time period. Choose a digit to represent radioactive nuclei that decay during this time period. This will yield a theoretical decay rate of 10 percent. Mark through and count each time this digit appears in your 500 digits. The number you mark out will be the number decayed during this first time period and will allow you to determine the undecayed number remaining after this time period.

4. Select a new block of digits for the undecayed number present that includes as many digits as the undecayed number remaining from above. Again, mark through all those that decay, and enter your results in the table.

5. Repeat this process until the table is complete or until fewer than 10 undecayed nuclei (i.e., digits) remain.

6. Make a graph of the number of experimental radioactive nuclei present vs. time. Once again, plot the number of radioactive nuclei present on the y-axis and the time on the x-axis. Draw a smooth curve through the data points.

From the August 2012 issue of MATHEMATICS teacher

Activity 1: Using Random Numbers to Simulate Radioactive Decay (continued)

Discussion Questions

1. Examine the data table entries and your graphs. Does the simulated data closely follow the theoretical predictions? Why or why not?

2. Suppose that someone asks you how much time it takes for 100 unstable nuclei to decay. How would you respond? Support your answer with data. (Suggestion: Examine the graph and see how long it takes for the sample to decay from 500 to 400 nuclei, from 250 to 150 nuclei, and so on.)

3. The half-life of a radioactive sample is the amount of time needed for one-half of all undecayed nuclei present to decay. According to your theoretical decay graph, the theoretical length of time necessary for your experimental number to go from 500 to 250 was approximately ___ time periods. If each time period represents 1 year, then your theoretical half-life would be ___ years.

4. Try this procedure for several other decay changes (from 400 to 200, from 100 to 50, and so on) and compare these with the length of time needed for the number of unstable nuclei to decay from 500 to 250.

5. According to these calculations, what is the approximate half-life of your radioactive sample?

6. Use your knowledge of exponential functions to calculate the theoretical half-life of the decaying random numbers.

Extension

Repeat this decay simulation using more than one digit to represent a decayed nucleus and compare these results with the previous results. What do you predict the half-life will be if two digits are used to represent decayed nuclei?

From the August 2012 issue of MATHEMATICS teacher

2014 Focus Issue

What Makes Ideas Stick?

Teachers design lessons to engage students with big ideas of mathematics. Does engagement necessarily imply that students will retain what we hope they will retain? How do students in your classes develop an appreciation for mathematical big ideas and the relationships among them? What lesson characteristics seem to promote the permanence of concepts in the minds of your students?

What works? Write about your big ideas—*big* big ideas or *small* big ideas—that help make mathematics stick. Following are a few suggestive, but not prescriptive, prompts:

- Great lessons stay in the minds of students. What lessons do you have that engage your students in exciting and meaningful mathematical experiences?
- Some ideas reside with students because of the instruction that introduces them more than the inherent mathematics. Do you start with an intriguing idea and then develop it throughout class, or do the big ideas build in the background?
- Students often surprise us with their own "aha!" moments, usually as they experience the epiphany of an unexpected connection. What are these moments, and how do we foster these connections and epiphanies?
- How do reasoning and sense making evolve to foster mathematical memory? What are high-leverage practices that help ideas stick?
- What kinds of challenges do big mathematical ideas face in the current political climate?
- How can technology impact the "stickiness" of ideas?
- How do we ensure that students internalize the mathematics that they need to be successful in subsequent courses? How do they develop the mathematical sophistication to adapt to an ever-changing world?

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Solutions to the Tipping Points Problem

TIPPING POINTS

Problem

The problem posed in *MT* August 2011 (vol. 105, no. 1, pp. 62–66) asked readers to consider the two-dimensional version of tipping a bowl (assumed to be a rectangular prism) to spoon out the last little bit of melted ice cream. Here is the essence of the problem: Given a fluid region of fixed area A contained in a rectangle whose width is W, find a formula for the fluid depth D when the container is tilted through a known angle T that is measured from horizontal.

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INVESTIGATION 1

Solution

When the container is not tilted, the fluid shape is a rectangle whose width W and height $H = A/W$ are known. The top segment of this rectangle is a fixed reference line. Let M denote its midpoint. Let $L = W/2$ denote the half width of the rectangle. When the water line tips slightly, the fluid region becomes a trapezoid.

Trapezoids will occur for a range of tilt angles, as indicated by the purple sector in **figure 1**. The figure shows that at a special tilt angle the shape of the fluid region changes from a trapezoid to a triangle. This tipping point, which marks the transition between two cases, occurs at the angle $c = \tan^{-1}(H/L)$. The formula for fluid depth D is a function that is defined piecewise by one formula in the trapezoidal case $T < c$ and another in the triangular case $T > c$.

The red diagonal of the half rectangle is another convenient auxiliary reference line from which we can measure angles. When the water line is above this diagonal red line, the fluid region is trapezoidal, but when the water line is below the diagonal, the fluid region is triangular. We discuss each case separately.

Case 1: Trapezoid

By symmetry, the top segment of the trapezoid must pass though the midpoint M of the length of the rectangle because the small triangles that lie above and below the horizontal reference line have equal areas. Thus, the area of the trapezoid through M equals the area of the original rectangle (see **fig. 1**).

The perpendicular distance from a tilted water line to the lower-left corner of the container can be found by trigonometry (see **fig. 2**). Consider the right triangle whose hypotenuse is the diagonal of the half rectangle. The hypotenuse length is $R = \sqrt{H^2 + L^2}$. The angle subtended at M, measured from the red diagonal to the water line, is the sum of the angle c (the tilt of the diagonal from the horizontal) and the angle T (the tilt of the water from the horizontal). The distance we seek is the length of the side opposite the angle whose measure is $\alpha = T + c$:

$$D = R\sin(T + c) \qquad (1)$$

This formula is valid only when the tilt angle is shallow: $0 < T < c$.

Case 2: Triangle

See **figure 3**. When the tilt angle T is steeper ($T > c$), the fluid region is a right triangle the length of whose hypotenuse will again be denoted by R. (This is not the same R as in case 1.) Labeling the other legs as $X = R\cos T$ and $Y = R\sin T$, the area of the triangle can be written as

$$Area_{\Delta} = \frac{XY}{2} = \frac{R^2 \sin T \cos T}{2} = \frac{RD}{2}$$

where D is the perpendicular distance from the hypotenuse to the opposite vertex. We can use these equations to learn that

$$R = \frac{D}{\sin T \cos T}.$$

We then eliminate R so that

$$Area_{\Delta} = \frac{D^2}{2\sin T \cos T} = \frac{D^2}{\sin(2T)}.$$

Thus, $D = \sqrt{\sin(2T)(Area)}$, which is a formula valid in the range of tilt angles

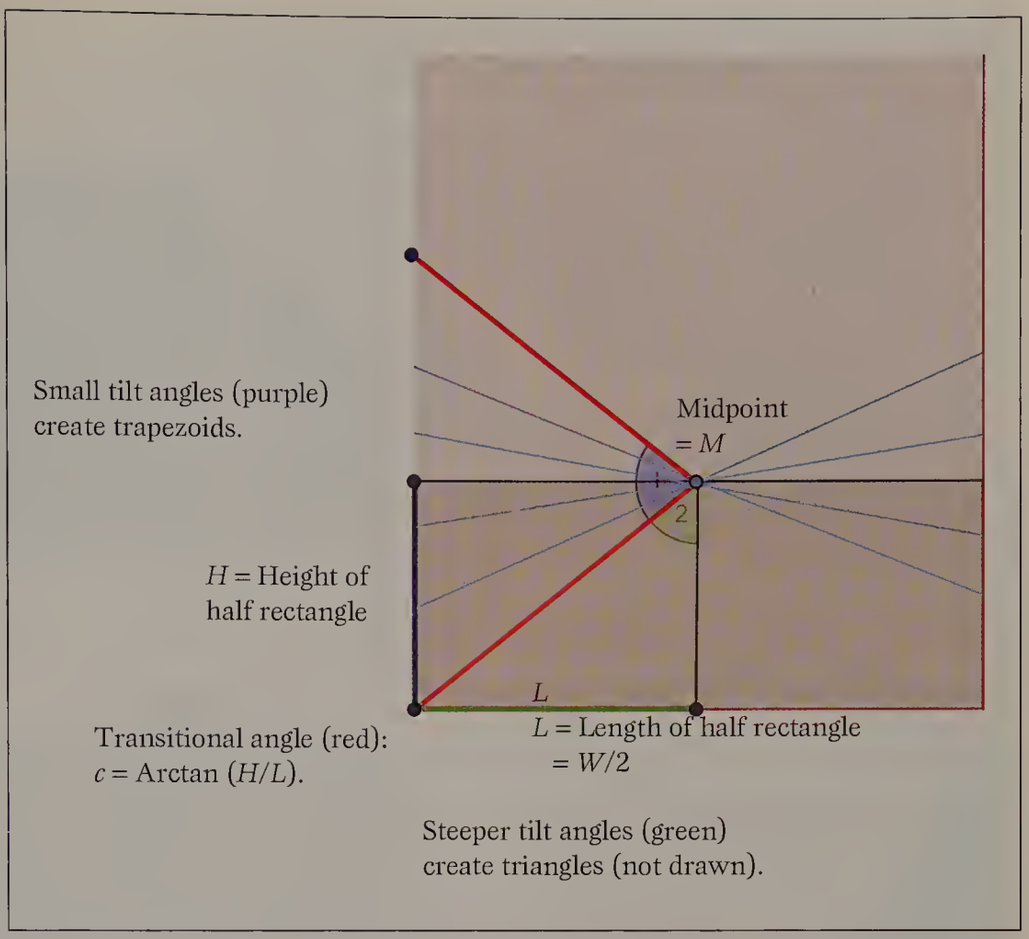
Small tilt angles (purple) create trapezoids.
Midpoint = M
1
2
H = Height of half rectangle
L
Transitional angle (red): c = Arctan (H/L).
L = Length of half rectangle = W/2
Steeper tilt angles (green) create triangles (not drawn).

Fig. 1 The tilt angle determines whether the fluid takes the shape of a trapezoid or a triangle.

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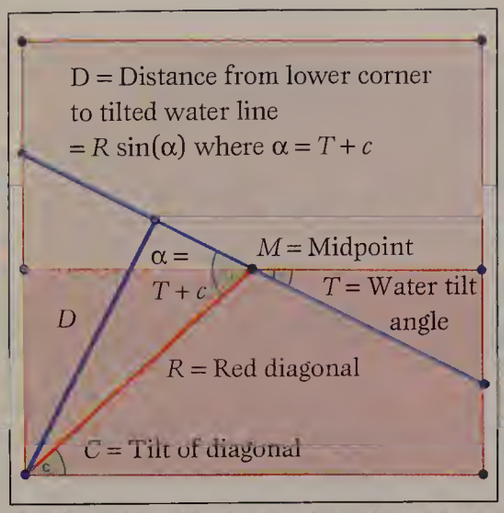

Fig. 2 Applying the distance formula yields the length of the perpendicular to the water line.

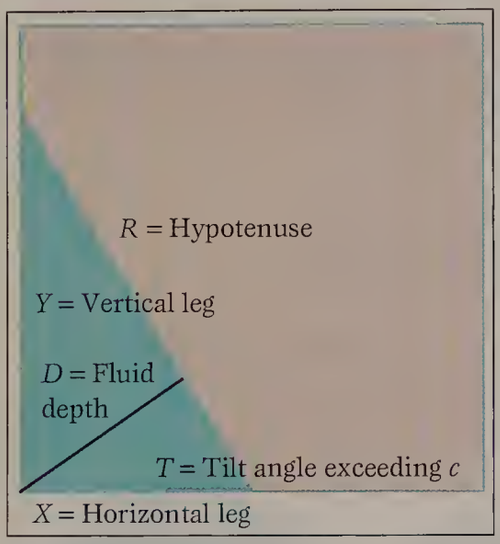

Fig. 3 The area of the right triangle can be related to the distance to the hypotenuse.

where $T > c$. Note that if we use the notation defined in case 1, then $Area = 2LH$. Thus,

$$D = \sqrt{2LH\sin(2T)} \quad \text{for } T > c = \tan^{-1}(H/L). \qquad (2)$$

INVESTIGATION 2: WHAT TILT ANGLE MAKES THE FLUID DEPTH *D* GREATEST?

Solution

The strategy for locating the optimal tilt angle depends on the dimensions H and L of the original fluid rectangle. The water depth, D, is a piecewise-defined function. It is described for small values of $T < c$ by equation (1) (part of a sine wave) and for large values of $T > c$ by equation (2) (the square root of a sine wave, whose graph resembles a semicircle). The two curves intersect when $T = c$ (see **fig. 4**).

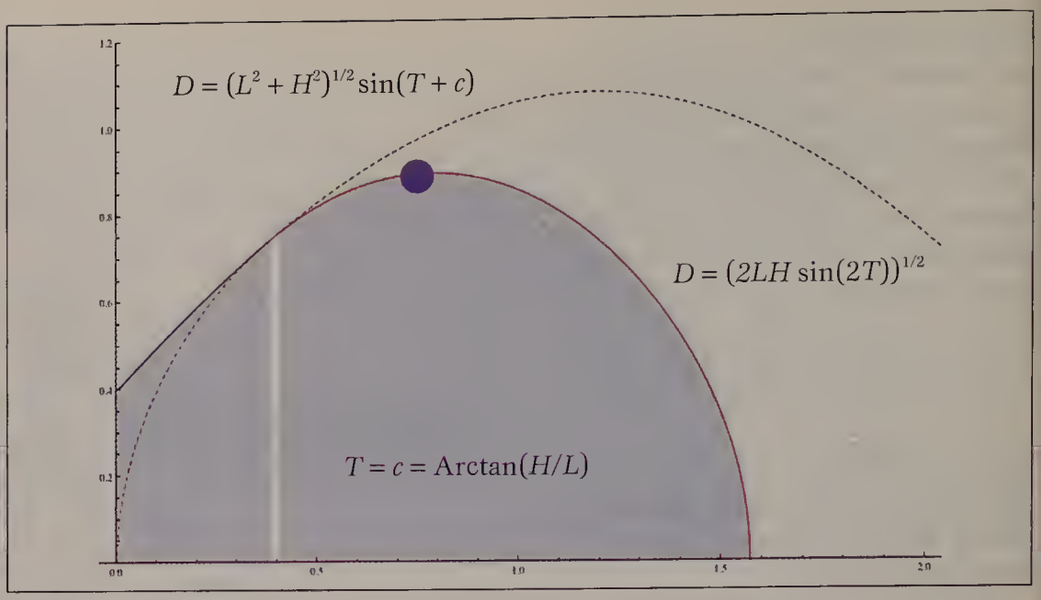

Fig. 4 Here, c is less than $\pi/4$

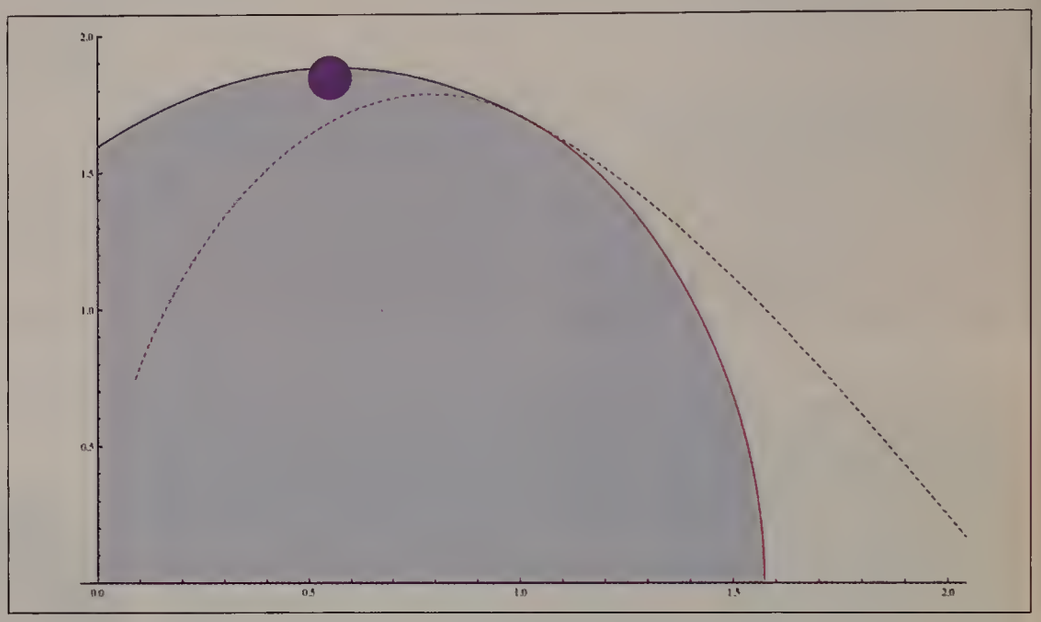

Fig. 5 The two curves intersect at $T = c > \pi/4$, and the maximum occurs to the left of c.

Case 1: Shallow Fluid Rectangles

If the fluid height satisfies $H < L$—that is, if the tipping point angle $c = \tan^{-1}(H/L)$ is smaller than $\pi/4$—then the optimal strategy is to tilt the fluid into the shape of an isoceles right triangle, tilted at the angle $T = \pi/4$. In case 1, the depth function $D = R\sin(T + c)$ increases as T increases for all $T < c$, until the tipping point is reached. Beyond this value, when $T > c$, the function

$$D = \sqrt{2LH\sin(2T)}$$

increases to its maximum at $T = \pi/4$.

Case 2: Deep Rectangles

On the other hand, if $H > L$, then the tipping point angle exceeds $\pi/4$ (see **fig. 5**). In this case, the optimal strategy is to tilt the fluid through the angle complementary to c: $T = \pi/2 - c$. With this strategy, the original fluid rectangle, if imagined to be a solid rigid body of ice frozen in place inside the container, would rotate so that the half rectangle's diagonal becomes vertical.

In case 2, the depth function $D(T) = R\sin(T + c)$ is initially an increasing sine function. The sine attains a maximum when $T + c = \pi/2$ and then declines as the tipping point $T = c$ is approached. Beyond this tipping point, the new description for depth is

$$D = \sqrt{2LH\sin(2T)}.$$

This is a decreasing function, since $T > \pi/4$, and therefore $2T > \pi/2$ is past the peak value for $\sin(2T)$.

Solve This Problem!

Locker Problem 1

A school has infinitely many lockers, numbered 1, 2, 3, 4, ... Each locker can be in one of two possible states—open or closed. Initially, the lockers are all closed. A prankster opens every locker. A second prankster closes every second locker: 2, 4, 6, ... The third prankster changes the state (an open locker is closed, a closed locker is opened) of every third locker: 3, 6, 9, ... The fourth prankster changes the state of every fourth locker: 4, 8, 12, 16, ... This process continues indefinitely, with the kth prankster changing the state of every kth locker. In the final state, which lockers will be open and which will be closed?

Locker Problem 2

Suppose instead that the second prankster changes the state of every fourth locker: 4, 8, 12, 16, ... The third prankster changes the state of every ninth locker: 9, 18, 27, ... The fourth prankster changes the state of every sixteenth locker: 16, 32, ..., and so on, with the kth prankster changing the state of the lockers k^2, $2k^2$, $3k^2$, ... What is the final state of locker 2000? (Hint: $2000 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5$)

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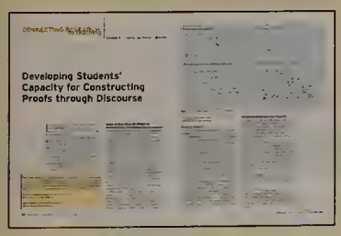
Linking research and practice has long been an NCTM strategic directive. NCTM's Research Committee has now advanced that goal with the Linking Research and Practice Outstanding Publication Award. Based on criteria ranging from timeliness to applicability, the annual award is given to a research-based article in one of the NCTM school journals. The 2011–2012 volume-year recipients are the following:



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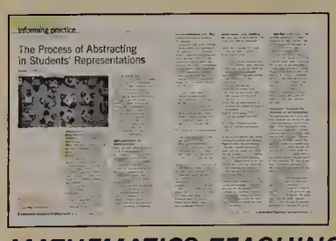
Despina A. Stylianou and Maria L. Blanton, "Connecting Research to Teaching: Developing Students' Capacity for Constructing Proofs through Discourse" (September 2011, pp. 140–45)



MATHEMATICS TEACHING IN THE MIDDLE SCHOOL

Despina A. Stylianou, "Informing Practice: The Process of Abstracting in Students' Representations" (August 2011, pp. 8–12)

TEACHING CHILDREN MATHEMATICS

P. Holt Wilson, Marrielle Myers, Cyndi Edgington, and Jere Confrey, "Fair Shares, Matey, or Walk the Plank" (April 2012, pp. 482–89)

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NCTM congratulates the recipients of this award, who will be acknowledged at the 2013 NCTM Research Presession and at the Annual Meeting and Exposition in Denver.

NCTM

Jon D. Davis

Investigating the Effects of Parameters on Quadratic Functions

Technology is in a constant state of flux. As a result, if we seek to use the latest forms of technology in our teaching, our teaching with technology must be ever changing. Edwards and Özgün-Koca (2009) describe an investigation involving the TI-Nspire CAS to understand the effect of b on the graphical representation of a quadratic function of the form $f(x) = ax^2 + bx + c$. Adapting this idea, I show how updates to technology can enhance this investigation to create an even more motivating and compelling experience for students. The investigation will make use of sliders and the spreadsheet capability of the TI-Nspire (the directions provided here apply to the TI-Nspire CX CAS).

SETTING UP SLIDERS

First, we will create three sliders to give us the ability to vary the values of a, b, and c interactively. Open a new document and create a **Graphs** page. Enter the function **f1(x)=a•x²+b•x+c**. Be sure to enter the multiplication symbol between the constant and the variable; otherwise, the software will interpret the variable differently. Insert a slider by selecting **menu**; **1: Actions**; and **A: Insert Slider**. A control box will appear on the screen (see **fig. 1**). Use the touchpad on the TI-Nspire to move the slider to a convenient place on the screen and click to set its position. The variable name for the slider will be highlighted. Type the letter **a** to name it and press **enter**. Follow the same steps to create sliders for variables b and c.

Once all three sliders for a, b, and c appear on the screen, move the cur-

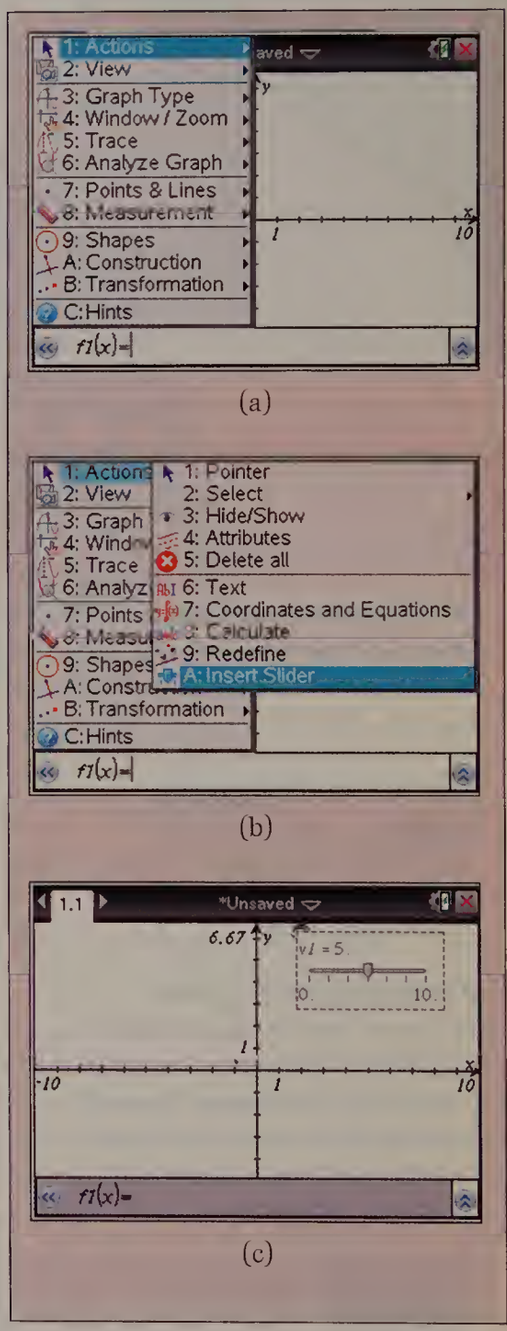

Fig. 1 Setting up a slider on the TI-Nspire CX CAS involves selecting the **menu** key and the **Actions** submenu.

Technology Tips, which provides a forum for innovative uses of technology in the teaching and learning of mathematics, appears seven times each year in *Mathematics Teacher*. Manuscripts for the department should be submitted via http://mt.msubmit.net. For more background information on the department and guidelines for submission, visit http://www.nctm.org/publications/content.aspx?id=10440#tech.

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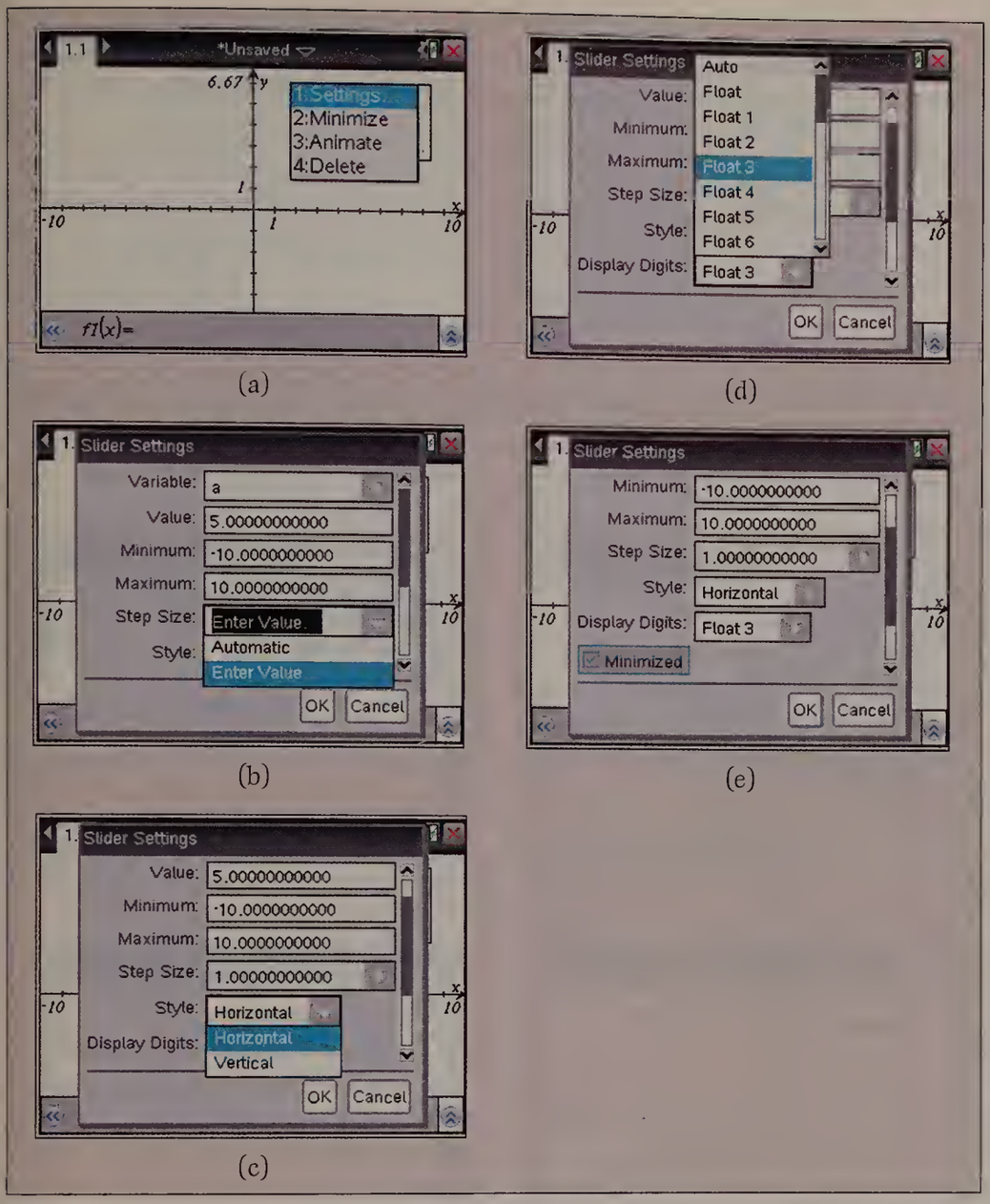

Fig. 2 These screen shots depict selecting the settings (**a**), changing the step size (**b**), choosing a horizontal style (**c**) with float 3 digits (**d**), and minimizing the slider (**e**).

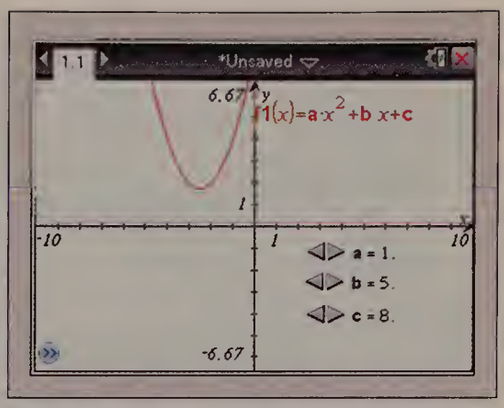

Fig. 3 Three sliders are now set up for each of the parameters in the general form of a quadratic function: $y = ax^2 + bx + c$.

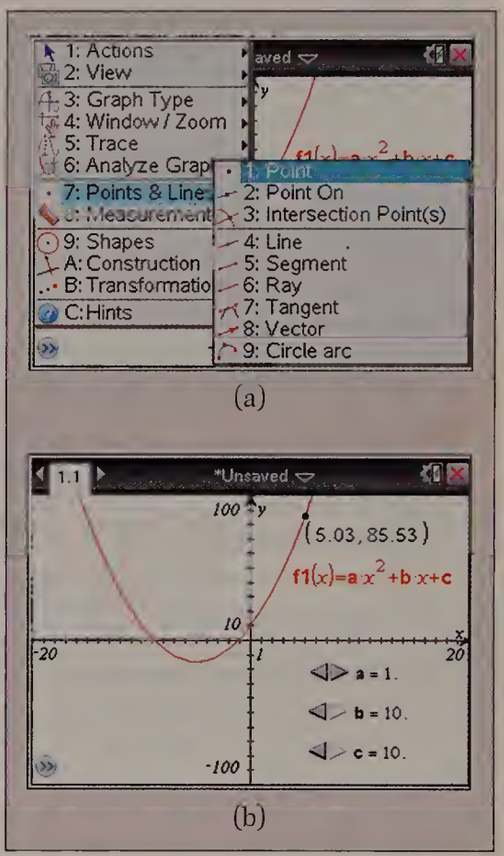

Fig. 4 A first step toward locating the maximum or minimum of the quadratic function requires placing a point on the graphical representation of the quadratic function.

sor over one of them and click on it so that the line surrounding the slider is solid instead of dashed. Push **ctrl** and then **menu**; then choose **settings**. Tab to the **Minimum** option and change it to **-10** (the **Maximum** should be **10**; if not, key in this value); change the **Step Size** to **Enter Value...**; select 1; select **Horizontal** for **Style** and **Float 3** for **Display Digits**; and then choose **Minimized** (see **fig. 2**). Repeat these steps for sliders b and c; the final view should be similar to that shown in **figure 3**.

To see the different graphical representations when the slider values change, we need to set the window to **XMin = -20, XMax = 20, XScale = Auto, YMin = -100, YMax = 100**, and **YScale = Auto**. These window settings can be set by choosing **menu** and then selecting **4: Window/Zoom** and choosing **1: Window Settings**. Select **OK** when finished.

To keep track of the vertex of the graphical representations as the sliders change, we will create a geometric construction on the coordinate plane. First, set the slider for a equal to **1** (move the cursor to the slider and select the correct arrow to change its value) and then set the sliders for b and c to 10. Select **menu**; **7: Points & Lines**; and **1: Point**. Then place a point on the graphical representation of the quadratic function just past $x = 5$ (see **fig. 4**). Press **esc** to exit the point tool.

Construct a line parallel to the x-axis through this point by selecting **menu**; **A: Construction**; and **2: Parallel**. Next, click on the point lying on the graphical representation of the quadratic function (5.03, 85.53), move down to the x-axis, and use the touchpad to construct a line parallel to the x-axis (see **fig. 5a**). Press **esc** to exit the parallel line tool. The lines that the TI-Nspire constructs do not extend across the screen, so you will need to drag the left end of the line by moving the cursor over it with the touchpad, selecting **ctrl**, and then pressing the touchpad. As you move the hand to the left with the pad, you should see the line grow in length. Press the hand key

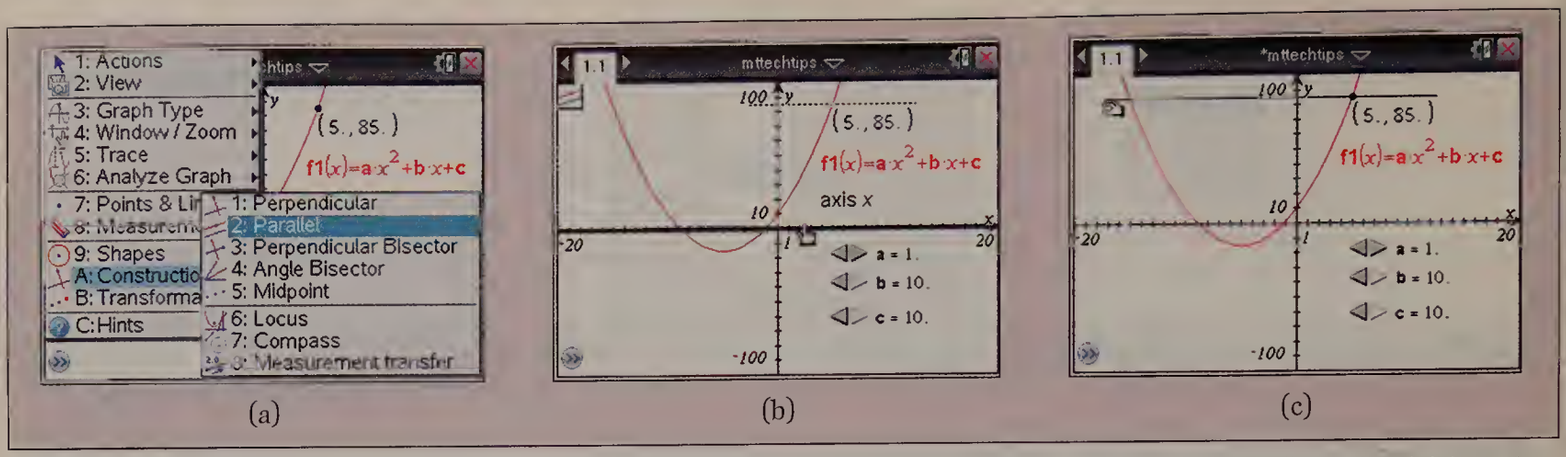

Fig. 5 We choose the construct parallel option (**a**), construct the line so that it is parallel to the *x*-axis (**b**), and extend the line so that it intersects the other side of the parabola (**c**).

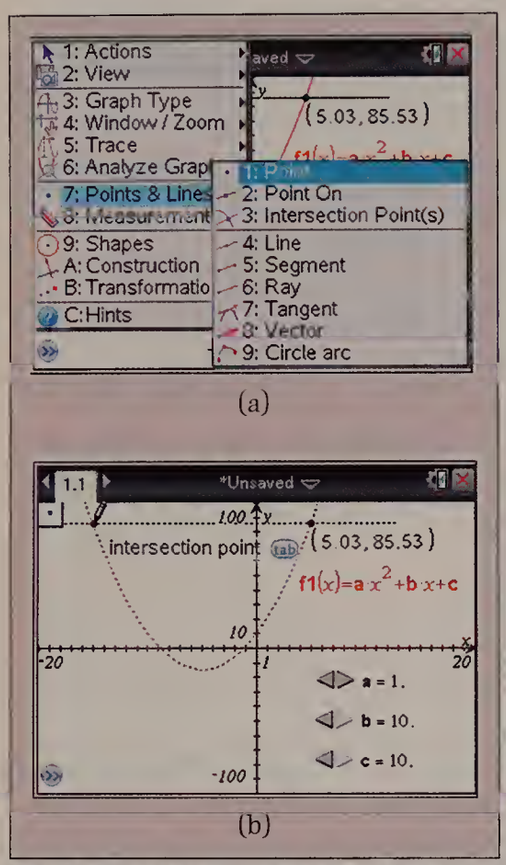

Fig. 6 We next place a point where the line intersects the curve.

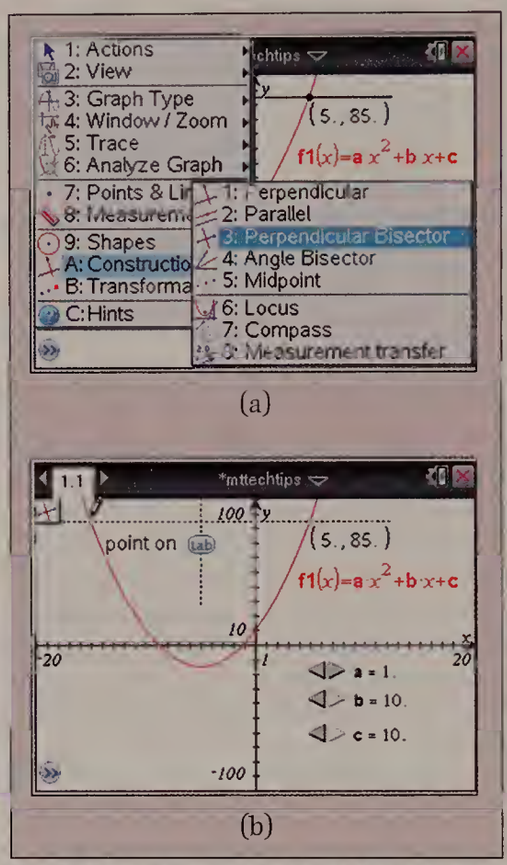

Fig. 7 These screen shots depict the creation of the perpendicular bisector of the segment spanning the parabola and parallel to the *x*-axis.

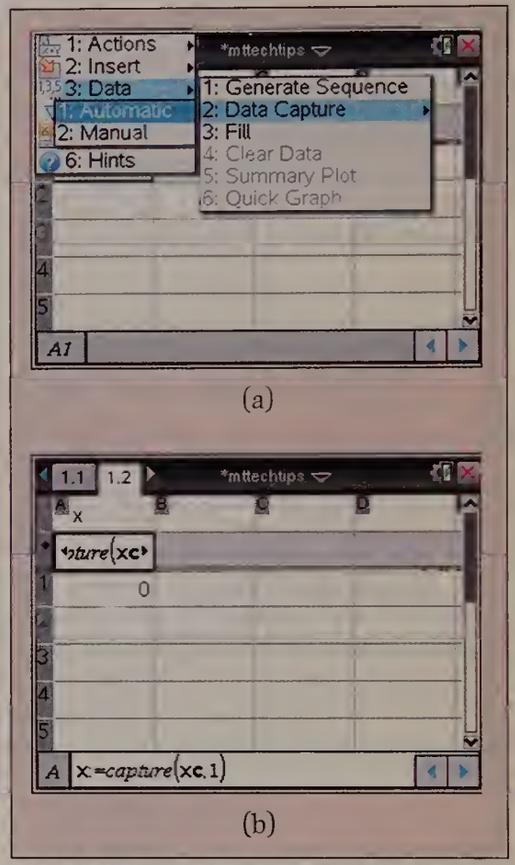

Fig. 8 Setting up the automatic data capture of the vertex of the parabola is accomplished within a **Lists & Spreadsheet** application.

to release the line once it intersects the other side of the graphical representation of the quadratic function (see **fig. 5**).

Place a point where this parallel line intersects the other side of the quadratic function using **menu**; **7: Points & Lines**; and **1: Point**. Move the cursor to the place where the line parallel to the *x*-axis intersects the other side of the quadratic equation and press the hand key (see **fig. 6**).

Construct the perpendicular bisector of the distance between the two points on the quadratic function by selecting **menu**; **A: Construction**; and **3: Perpendicular Bisector**. Click on one of the points where the line parallel to the *x*-axis intersects the graphical representation of the quadratic function. Move to the other point on the parallel line and click on the point (see **fig. 7**). Press **esc** to exit the perpendicular bisector tool.

Just as you dragged the end of the parallel line so that it intersected the quadratic function, we need to follow a similar process here so that the perpendicular bisector passes through the vertex of the quadratic function. Place a point at the intersection of the perpendicular bisector and the vertex of the quadratic function, again using the **Point** command. Next, display the coordinates on the screen by selecting **menu**; **1: Actions**; and **7: Coordinates and Equations**. Then click on the intersection point. Move the coordinates to a convenient place on the screen and click on them to set their position.

GRAPHING VERTICES AS PARAMETERS VARY

Using the selection tool, move the cursor over the *x*-coordinate of the vertex until an open hand appears and click on it. Press **ctrl** and then **sto/var**; then type

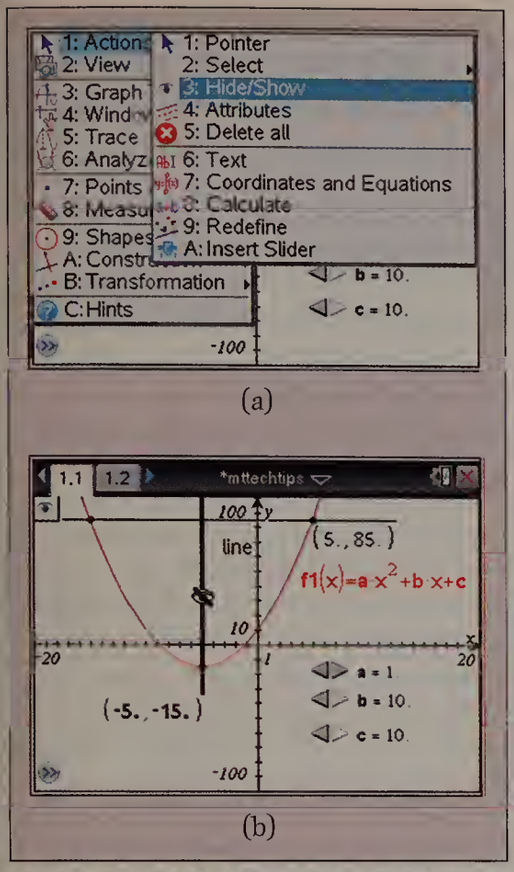

Fig. 9 Use the **Hide/Show** tool to hide elements of the construction that are no longer needed.

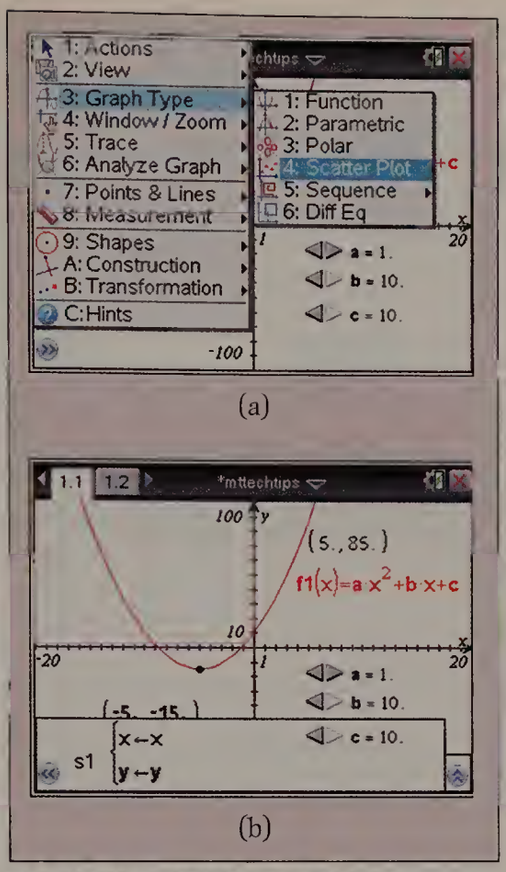

Fig. 10 The vertices for the different quadratic functions that were captured earlier can be graphed as a scatter plot.

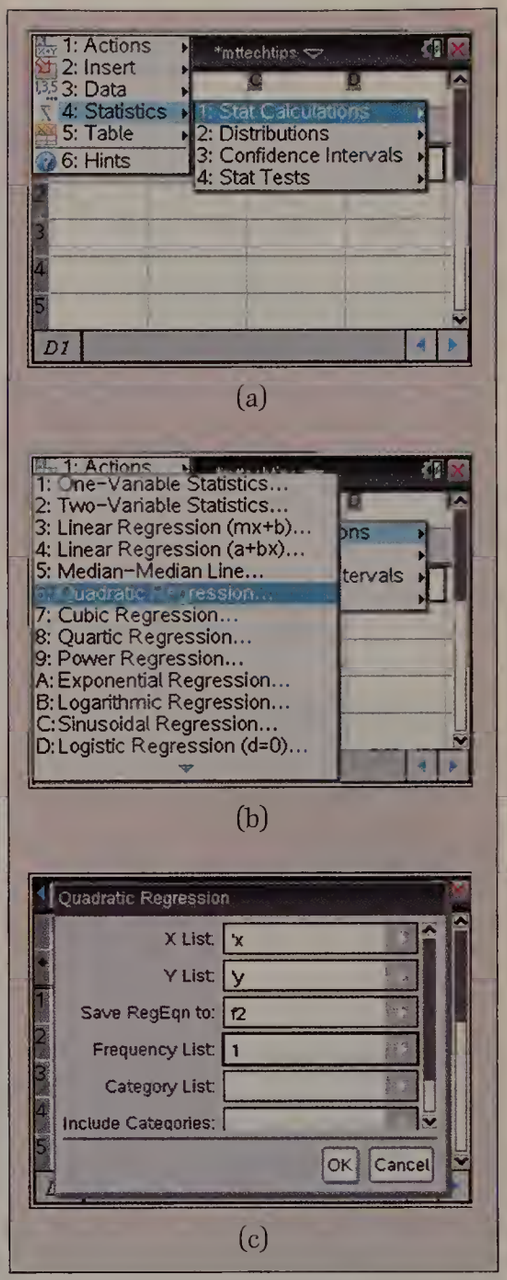

Fig. 11 Students create a quadratic regression model for the vertices.

xc and select **enter** to store this value in a variable called **xc**. The *x*-coordinate of the vertex should now appear in bold. Repeat this process to store the *y*-coordinate of the vertex as the variable **yc**. Insert a **Lists & Spreadsheet** application using **ctrl-I**. To label the first column, type **x** in the naming cell (the top cell). We will use the automatic data capture feature of the calculator to enter the coordinates of the vertices into the spreadsheet. Move to the formula cell (gray with the diamond) and select **menu**; **3: Data**; **2: Data Capture**; and **1: Automatic.** Enter **xc** (see **fig. 8**). Repeat the process, labeling the second column **y** and using data capture for the value of **yc**.

Go back to the previous page in the document by selecting **ctrl** and the left arrow on the touchpad. We can hide extraneous features of the construction by selecting **menu**; **1: Actions**; and **3: Hide/Show.** Move the cursor over the perpendicular line and click on it to hide it. The perpendicular line will be much dimmer but can still be seen. Continue by hiding the line parallel to the *x*-axis and the intersection points. When finished, press **esc,** and all those objects will be hidden from view (see **fig. 9**).

To create a graphical representation for the coordinates of the vertices as the parameter for *b* changes, change the graphing mode to a scatter plot by selecting **menu**; **3: Graph Type**; and **4: Scatter Plot**. At the prompt for *x*, press the **var** key and chose variable **x**. Use the touchpad to go down to the space for *y*, type **y**, and select **enter** (see **fig. 10**). Now use the slider to vary the parameter *b* in the quadratic function through the full range of possible values. The coordinates of the vertices will appear on the screen.

FINDING A SYMBOLIC REPRESENTATION FOR THE VERTEX SCATTER PLOT

Students can use a number of different methods to locate the symbolic representation for the path taken by the vertices. For instance, students could enter the function **f2(x)=-x²** and manually manipulate its graphical representation until it fits the points of the scatter plot, as noted by Edwards and Özgün-Koca (2009). Unfortunately, many students use the regression capabilities of the calculator to find a symbolic representation for a set of data without first thinking about the mathematical structure underlying the data. Zbiek (1998) described how prospective teachers used the goodness-of-fit values generated by the technology without examining a graph of the symbolic representation of the function with the scatter plot. Students also failed to consider other factors such as the context of the real-world situation being modeled.

To counter this tendency, teachers could ask students to think about what function might fit the path of the vertices and why, suggesting that they examine the graphical representation as well as the tabular representation. The regression feature is valuable in this investigation because the Nspire will automatically locate the regression

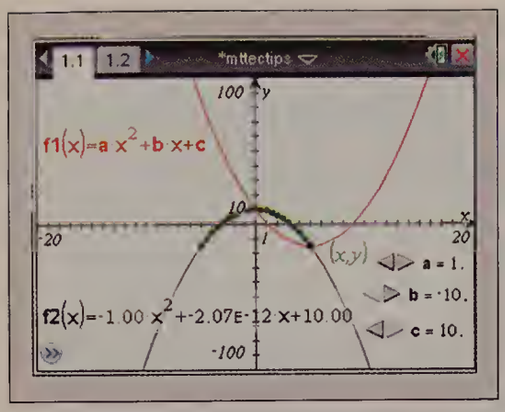

Fig. 12 This screen shot shows the quadratic regression function $f2(x) = -x^2 + 10$ for the set of vertices produced when b varies from -10 to 10, $a = 1$, and $c = 10$.

equation as slider values change and the vertices take different paths.

Return to the **Lists & Spreadsheet** page (select **ctrl** and then →), which contains all the x- and y-coordinates for the quadratic function vertices. Move to cell C1 in the spreadsheet and select **menu**; **4: Statistics**; **1: Stat Calculations**; and **6: Quadratic Regression** (see **fig. 11**). Move the x-variable into the **X List** by clicking on the arrow in the box for **X List** and selecting **x**. Follow a similar process to place the y-variable. Then tab through the rest of the options and click on **OK**.

Our work reveals that the quadratic function that fits the vertices when $a = 1$ and $c = 10$ is $f(x) = -x^2 + 10$. Let's graph this function just to be sure that it does indeed fit the scatter plot of the vertices. Go back to the **Graphs** page (select **ctrl** and then ←) and select **menu**; **3: Graph Type**; and **1: Function**. In the entry line, find the quadratic regression stored in the function $f2(x)$ by scrolling up (use the touchpad to move up) and press **enter** to obtain its graph (see **fig. 12**). At this point, it is a good idea to save the document that has been created. Select **ctrl**, followed by the letter **S**; then type in a name for the document and select **enter**.

Students should continue to examine how b influences the quadratic function by setting a to a different value, leaving c fixed, and varying b. The current graph can continue to be used. Use the slider to set $a = 2$. Next, go into the **Lists & Spreadsheet** page and reset the automatic data collection by moving to the cell that contains **x:=capture(xc,1)**. After highlighting the cell, hit **enter** twice. A window reporting a dimension error

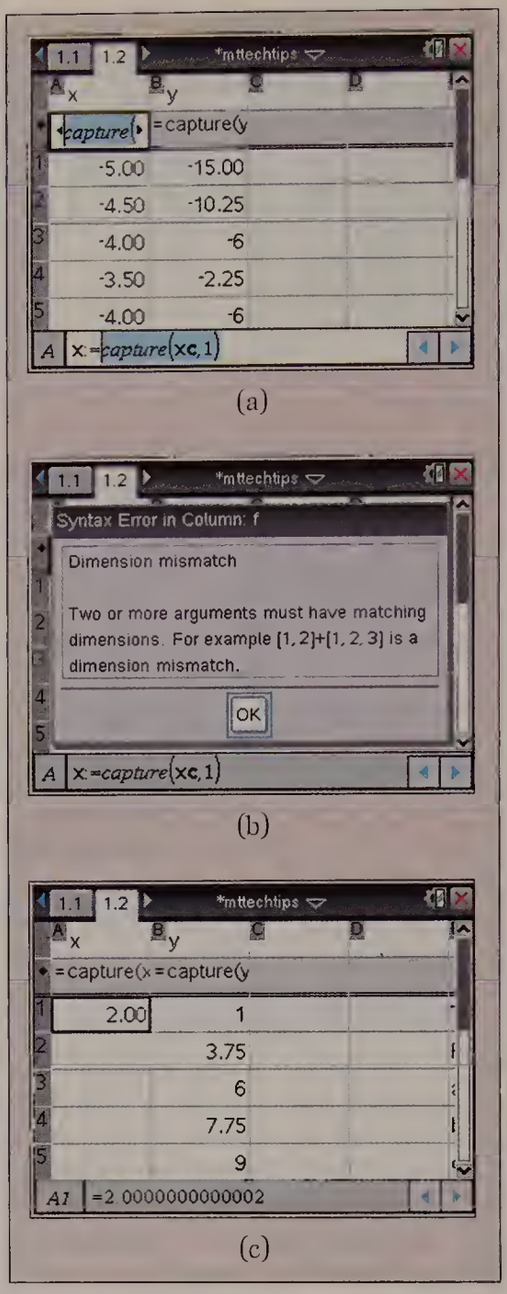

Fig. 13 Selecting the cell below the column heading and **enter** (**a**) will give us a dimension mismatch error (**b**), but choosing **enter** one more time will clear the column so that we can repeat the experiment with different values (**c**).

appears, but we can ignore this by hitting **enter** (see **fig. 13**). Follow a similar procedure for the other cell that contains **y:=capture(yc,1)**.

We can now go back to the page with the graph and vary b by using the slider with a different set of a and c values. A different quadratic regression equation should now appear for these points. Now we can investigate how the graph of the quadratic function moves when b varies for a different set of a and c values. We simply set a or c (or both) to the values of our choosing, return to the **Lists & Spreadsheet** application in the next page, follow the steps in **fig. 13**, and go back to the graphs application to vary b and capture another quadratic regression for that set of vertices.

CONCLUSION

This investigation, building on and extending earlier ideas, allows students to investigate the influence of b on the graphical representation of quadratic functions other than those where $a = 1$ and $c = 2$. Before the advent of technology equipped with sliders and dynamic, linked representations, this investigation was possible but extremely time-consuming. Moreover, the amount of work required to draw the graphs by hand or create multiple graphs on a standard graphing calculator could easily cause students to miss the forest for the trees. This investigation gives students a chance to focus on the influence of b on quadratic graphical representations without needing to attend to the details of constructing each graphical representation.

When working with functions, students often focus on independent variables and their associated dependent variables. This activity provides students with a deeper understanding of the quadratic function family and an opportunity to develop a more sophisticated view of functions by considering parameters as inputs and a specific quadratic function as the output.

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SURFING NOTE

The website http://mathsci2.appstate.edu/~sjg/simpsonsmath/, maintained by Sarah Greenwald and Andrew Nestler, includes an index of the more than one hundred mathematical references made on the animated television series *The Simpsons* during the twenty years that the show has been aired. The index also includes links to activity sheets that Greenwald and Nestler have used to connect these references to meaningful mathematical activities with students.

simpsonsmath.com

The Simpsons has established itself as an award-winning international pop culture phenomenon. It is the longest-running sitcom of all time and it is also one of the most literate television programs on the air, containing many references to subject matter and scholars from various academic fields, including mathematics. Since *The Simpsons* has been airing in prime-time for most of our students' lives, they likely are familiar with the program and its large cast of characters, including a resident scientist. *The Simpsons* also contains over a hundred instances of mathematics ranging from arithmetic to geometry to calculus, many designed to expose and poke fun at innumeracy. In fact, Al Jean, Executive Producer and head writer, has a bachelor's degree in mathematics from Harvard University. Several episodes of *The Simpsons* contain significant mathematics that relates to material we normally cover in our classes. For these reasons, this program is an ideal source of fun ways to introduce important concepts to students, and to reduce math anxiety and motivate students in courses for non-majors.

Dr. Sarah J. Greenwald, Appalachian State University, Boone, NC
Dr. Andrew Nestler, Santa Monica College, Santa Monica, CA

You may remember us from such talks as "The Simpsons Rule: Mathematical Morsels from *The Simpsons*" (aimed at a general audience) and "r dr r: Engaging Students with Significant Mathematical Content from *The Simpsons*"

Mathematics on The Simpsons

Guide to Mathematics and Mathematicians on *The Simpsons*

Mathematical Backgrounds of *The Simpsons*' Writers

MoneyBart original airdate 10/10/10

Girls Just Want to Have Sums original airdate 4/30/06

See also mathematics on *Futurama*

Engaging Students with Related Mathematics

Wizard of Evergreen Terrace

Classroom Activity Sheets

Text transcript of our 15-minute presentation "r dr r: Engaging Students with Significant Mathematical Content from *The Simpsons*" and the related PRIMUS article, Vol XIV, Num 1, March 2004, pp. 29-39. See also impact on students

Summary of Math-Club whose audience included a number of *Simpsons* and *Futurama* writers, April 6, 2005

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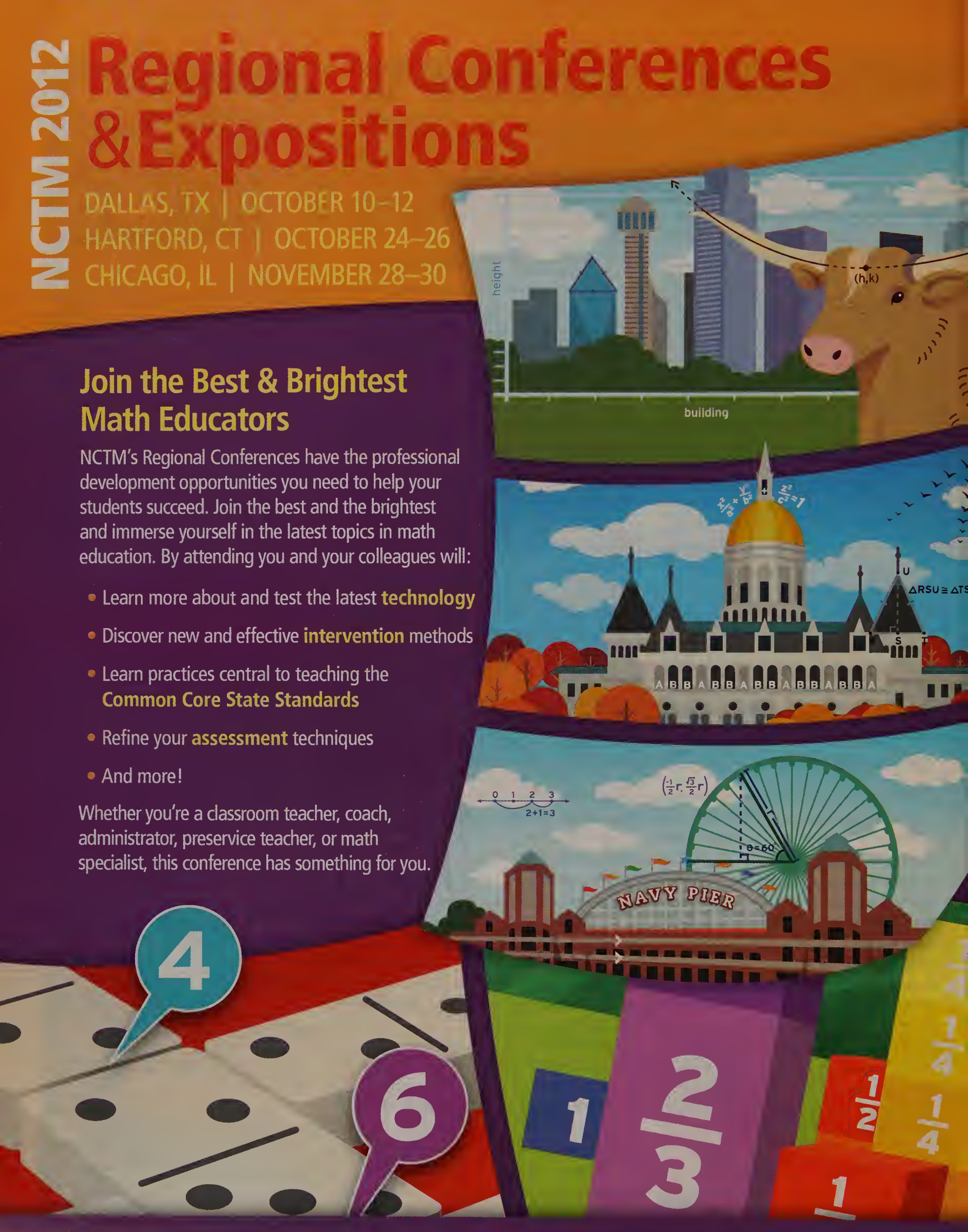

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Dohyoung Ryang and Tony Thompson

Sums of Positive Integral Powers

The sum of the first n positive integers $1 + 2 + 3 + \cdots + n = \Sigma k$ and the sum of the first n squares $1^2 + 2^2 + 3^2 + \cdots + n^2 = \Sigma k^2$ arise in several middle school and high school mathematics problems—for example, in the handshake and cannonball problems (e.g., Driscoll 1999). The solutions involve familiar formulas expressing each sum as a function of n:

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

and

$$1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}.$$

When we discuss the handshake and cannonball problems with other teachers as well as with high school and college students, one question that often comes up is whether formulas can be found for the sum of consecutive cubes and higher powers. The question can be stated in general terms as follows: "For a natural number t, what formula expresses the sum $\Sigma k^t = 1^t + 2^t + 3^t + \cdots + n^t$ as a function of n?" We present one approach to investigating such sums using methods accessible to high school and college teachers and their students. (Throughout, summations denoted $\Sigma f(k)$ will be assumed to run as k goes from 1 to n.)

A REVIEW OF TWO FAMILIAR PROBLEMS

The Handshake Problem

First, let's review the handshake problem, a typical version of which is shown in **figure 1**. Readers probably know that the answer to this problem can be formulated in terms of a sum of consecutive integers, as follows. With two attendees, there is only one handshake. When a third attendee arrives, the new guest must shake hands with two people, so there is a total of three handshakes. When the fourth guest arrives, he or she must shake hands with each of the three guests already in the room, and when the ith person enters the room, that guest must shake hands with all the $i - 1$ guests previously in the room. So with $n + 1$ people at the party, the number of handshakes equals the sum $1 + 2 + \cdots + n = \Sigma k$ and, as noted before, is expressed as a function of n by $\Sigma k = n(n + 1)/2$.

We can obtain this result in several ways. One involves writing the sum twice, running from 1 to n in one instance and backward from n to 1 in the other:

Delving Deeper offers a forum for classroom teachers to share the mathematics from their own work with the journal's readership; it appears in every issue of *Mathematics Teacher*. Manuscripts for the department should be submitted via http://mt.msubmit.net. For more background information on the department and guidelines for submitting a manuscript, visit http://www.nctm.org/publications/content.aspx?id=10440#delving.

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$$\sum k = 1+2+3+\cdots+(n-2)+(n-1)+n$$
$$\sum k = n+(n-1)+(n-2)+\cdots+3+2+1$$

These lines add up to $2\Sigma k$. On the right side of the equation are n columns, each of which totals $n + 1$. Thus, $2\Sigma k = n(n + 1)$ or $\Sigma k = n(n + 1)/2$. For example, the sum of the first 100 natural numbers is $(100 \cdot 101)/2 = 5050$.

The formula for Σk can also be obtained by using combinations. There is one handshake for two individuals at the party. If we number the partygoers, each pair can be represented by the form (i, j), where we do not distinguish between (i, j) and (j, i). So the number of handshakes is equal to the number of combinations when we choose two distinct objects from a set of $n + 1$. By the definition of combination, we have

$$_{n+1}C_2 = \frac{(n+1)!}{2!(n-1)!} = \frac{n(n+1)(n-1)!}{2(n-1)!} = \frac{n(n+1)}{2}.$$

The Handshake Problem

At a party I attended recently, I noticed that every person shook hands with each other person exactly one time. There were 20 people at the party.

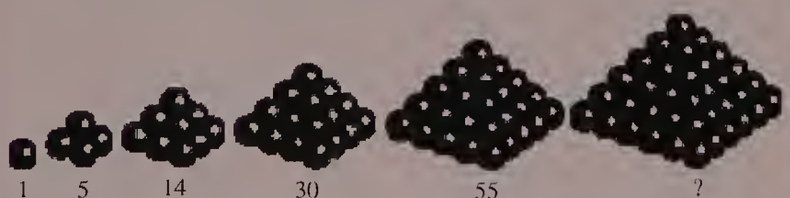
1. How many handshakes were there?
2. What is the number of handshakes if 100 people attended the party?
3. Write a function to describe the pattern for the number of handshakes for $n + 1$ people.

Adapted from Driscoll (1999)

Fig. 1 The handshake problem, which can be solved in several different ways, is considered here as a sum of consecutive integers.

The Cannonball Problem

One way to stack cannonballs is to form a pyramid with a square base. The diagram below portrays the cumulative number of balls for the six smallest pyramids. In each pyramid, the layers are square arrangements of balls. The top layer is a single ball; the next layer down is a 2×2 square; the next layer a 3×3 square; and so on. As suggested by the diagram, the first pyramid consists of 1 cannonball, the second pyramid consists of 5 cannonballs, the third consists of 14 cannonballs, and so on.



1. How many cannonballs make up the sixth pyramid? The tenth pyramid?
2. Find a function that describes the relationship between the number of layers in the pyramid and the number of cannonballs in the pyramid.

Adapted from Driscoll (1999)

Fig. 2 The cannonball problem is not as well known as the handshake problem.

The Cannonball Problem

In addition to the handshake problem, some high school students are likely to have encountered a version of the cannonball problem (see **fig. 2**). Because each layer of the cannonball pyramid is a square, we can see that the top layer consists of 1 ball, the second layer consists of 2^2 balls, the third layer consists of 3^2 balls, and so on. Thus, the number of cannonballs in an n-layered pyramid equals Σk^2. This number can also be expressed as $\Sigma k^2 = n(n + 1)(2n + 1)/6$, as noted earlier.

We have several ways of finding this expression and applying it to other problems (e.g., Greive 1999; Hegblom 1993; Copper 1986; Klinger 2006; Nelson 1993), and many readers may be aware that the formula for Σk^2 can be found using the STAT function on a graphing calculator. However, our earlier derivations of the formula for Σk do not generalize in an obvious way to the case of Σk^2.

GENERALIZING SUMS

Returning to the question posed earlier: Do functional expressions exist for Σk^t where t is a positive integer? For the purpose of generalization, we observe that $\Sigma k^0 = \Sigma 1 = n = n/1$. Combined with the previous results, this produces a suggestive sequence of formulas:

$$\sum k^0 = \frac{n}{1}$$
$$\sum k^1 = \frac{n(n+1)}{2}$$
$$\sum k^2 = \frac{n(n+1)(2n+1)}{6}$$

From this sequence, teachers and students often see possible patterns for sums of cubes and fourth-higher powers. For example, one common conjecture is that the number of binomial factors increases by 1 in each successive formula. In particular, the final binomial factor in the formula for Σk^t should be $(tn + 1)$. Because the denominators are $1 = 1!$, $2 = 2!$, and $6 = 3!$, we would naturally guess that the next is $24 = 4!$. Combining both observations, we might conjecture that $\Sigma k^3 = n(n + 1)(2n + 1)(3n + 1)/24$. It is easy to verify that this formula fails very quickly.

Although the formulas for Σk^t do contain valid patterns, they are not easy to find using only the three examples given above. We might at least guess that the formula for Σk^t is a polynomial in n of degree $t + 1$ and then find formulas for several values of t greater than 2 using methods of polynomial interpolation. But it turns out that there is an insightful method for analyzing sums of the form Σk^t directly. It is based on the idea of *telescoping sums*.

The key idea is embodied in the formula $\Sigma(a_k - a_{k-1}) = a_n - a_0$, meaning that the sum of differences between consecutive terms in a sequence is the last term minus the initial term. We can see why this equation holds by writing out the pattern of terms but in reverse order:

$$\sum\left(a_k - a_{k-1}\right) = \left(a_n - a_{n-1}\right) + \left(a_{n-1} - a_{n-2}\right) + \left(a_{n-2} - a_{n-3}\right) + \cdots + \left(a_1 - a_0\right)$$

This sum gets its name because all terms between the first and last add to zero, thus causing the sum to collapse like a telescope (see **fig. 3**). We illustrate the use of telescoping sums by returning to our formulas for Σk^t for the cases $t = 1$ and $t = 2$.

For the case in which $t = 1$, we consider differences for the sequence $a_n = n^2$. We know that the formula will be quadratic, so we start with the identity $1 + 2k + k^2 = (1 + k)^2$ (and remark for future reference that this identity is an application of the binomial theorem with exponent 2). Moving the k^2 term to the right side of the equation produces the difference of successive squares: $1 + 2k = (1 + k)^2 - k^2$. Summing over k, we have the following:

$$\sum\left(1 + 2k\right) = \sum\left(\left(1 + k\right)^2 - k^2\right)$$

We then use properties of sums on the left and the telescoping sum formula on the right to obtain

$$\sum 1 + 2\sum k = \left(1 + n\right)^2 - 1.$$

To simplify further, we observe that $\Sigma 1 = n$. Thus, we can solve $n + 2\Sigma k = (1 + n)^2 - 1$ for Σk to obtain

$$\sum k = \frac{\left(1 + n\right)^2 - 1 - n}{2} = \frac{n^2 + n}{2} = \frac{n\left(n + 1\right)}{2}.$$

We can extend this method to the case in which $t = 2$. We again begin by applying the binomial theorem, now with exponent 3, writing $1 + 3k + 3k^2 + k^3 = (1 + k)^3$. Repeating the same steps as before, we rearrange this equation as $1 + 3k + 3k^2 = (1 + k)^3 - k^3$ and sum on k to find that

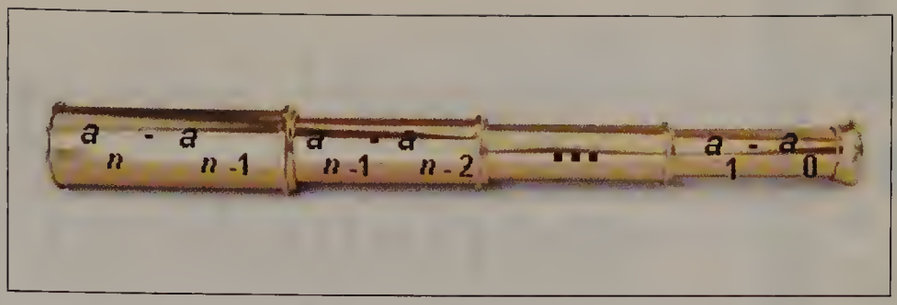


Fig. 3 An old-fashioned telescope gives its name to certain sums.

$$\sum\left(1 + 3k + 3k^2\right) = \sum\left(\left(1 + k\right)^3 - k^3\right).$$

Continuing as before, we use properties of sums on the left and the telescoping sum identity on the right, producing $\Sigma 1 + 3\Sigma k + 3\Sigma k^2 = (1 + n)^3 - 1$. At this point in the previous derivation, we replaced $\Sigma 1$ with n. This time we also replace Σk with $n(n + 1)/2$. Then, from

$$n + \frac{3n(n+1)}{2} + 3\sum k^2 = (1 + n)^3 - 1$$

we can solve for Σk^2 to obtain

$$\sum k^2 = \frac{(1+n)^3 - 1 - n}{3} - \frac{n^2 + n}{2} = \frac{n(n+1)(2n+1)}{6}.$$

It should now be clear how to proceed to higher values of t. We invite readers to derive the identity $\Sigma k^3 = n^2(n + 1)^2/4$.

Although we can continue as above to derive formulas for the next several Σk^t, we are now able to generalize. To find Σk^t, we proceed as before with the binomial expansion of $(1 + k)^{t+1}$. To simplify the notation, let $s = t + 1$. The binomial expansion that we need is then

$$\left(1 + k\right)^s = {}_sC_0 + {}_sC_1 \cdot k + \cdots + {}_sC_{s-1} \cdot k^{s-1} + {}_sC_s \cdot k^s$$

where we assume that $s \geq 2$. Noting that ${}_sC_s = 1$ and ${}_sC_{s-1} = s$, we rewrite the equation as

$$\left(1 + k\right)^s - k^s = {}_sC_0 + {}_sC_1 \cdot k + \cdots + {}_sC_{s-2} \cdot k^{s-2} + sk^{s-1}$$

and sum over k. The left side telescopes, leaving us with

$$\left(1 + n\right)^s - 1 = {}_sC_0 \cdot \sum 1 + {}_sC_1 \cdot \sum k + \cdots + {}_sC_{s-2} \cdot \sum k^{s-2} + s\sum k^{s-1}.$$

Solve for Σk^{s-1} to get

$$\sum k^{s-1} = \frac{1}{s}\left[\left(1+n\right)^{s} - 1 - {}_{s}C_{0} \cdot \sum 1 - {}_{s}C_{1} \cdot \sum k - \cdots - {}_{s}C_{s-2} \cdot \sum k^{s-2}\right]$$

and then reintroduce $t = s - 1$ to reach the identity

$$\sum k^{t} = \frac{1}{t+1}\left[\left(1+n\right)^{t+1} - 1 - {}_{t+1}C_{0} \cdot \sum 1 - {}_{t+1}C_{1} \cdot \sum k - \cdots - {}_{t+1}C_{t-1} \cdot \sum k^{t-1}\right].$$

This equation, which is true for all positive integers t, can be expressed more compactly by defining $P_t(n)$ as Σk^t and using sigma notation:

$$P_t(n) = \frac{1}{t+1}\left[\left(1+n\right)^{t+1} - 1 - \sum_{i=0}^{t-1} {}_{t+1}C_{i} \cdot P_i(n)\right] \quad (1)$$

To demonstrate this formula, let's use it with $t = 1$ to find Σk:

$$P_1(n) = \left(\frac{1}{1+1}\right)\left[\left(1+n\right)^{1+1} - 1 - \sum_{i=0}^{1-1} {}_{1+1}C_{i} \cdot P_i(n)\right]$$
$$= \left(\frac{1}{2}\right)\left[\left(1+n\right)^{2} - 1 - {}_{2}C_{0} \cdot P_0(n)\right]$$
$$= \left(\frac{1}{2}\right)\left[\left(1+n\right)^{2} - 1 - 1 \cdot n\right] = \frac{n(n+1)}{2}$$

Not surprisingly, these manipulations are similar to the earlier telescoping sum derivation. A similar observation occurs when we set $t = 2$ in (1). Readers who earlier applied the telescoping sum method for Σk^3 may now wish to compare that derivation to the result of setting $t = 2$ in identity (1). As a final example, we state the result for $t = 4$. Without showing all the steps, we see that identity (1) leads to

$$P_4(n) = \left(\frac{1}{5}\right)\left[\left(1+n\right)^{5} - 1 - \sum_{i=0}^{3} {}_{5}C_{i} \cdot P_i(n)\right]$$
$$= \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.$$

OBSERVATIONS TO BE MADE

In identity (1), $P_t(n)$ depends on the values of all $P_j(n)$ with $j < t$. Is it possible to obtain $P_t(n)$ by just using $P_{t-1}(n)$? As a first step toward answering this question, let's consider again the formulas for $P_j(n)$ that have already been found, expressing each in both factored and descending form (see **table 1**). Take a moment and consider what observations you or your students can make regarding these formulas.

Five of our observations follow:

1. Each $P_j(n)$ is a polynomial of degree $j + 1$.
2. The leading coefficient in each $P_j(n)$ is $1/(j + 1)$.
3. The second coefficient in each $P_j(n)$ is $1/2$.
4. In each case, $P_j(n)$ has no constant term.
5. For each $P_j(n)$, the sum of the coefficients is 1.

Although these observations are purely empirical, they can be verified to hold for all $P_j(n)$ by using identity (1) and induction. The last two observations follow directly from the definition of $P_j(n)$. For $n = 0$, $P_j(n)$ is the sum of no terms and so is equal to 0. For $n = 1$, $P_j(n)$ is the single term $1^j = 1$. We also know that for any polynomial p, $p(0)$ equals the constant term and $p(1)$ equals the sum of the coefficients. Thus, for $P_j(n)$ the constant term is 0 and the sum of the coefficients is 1.

Observation 2 allows us to compute the leading coefficient directly for any j, but a recursive approach is also possible. For example, the first two leading coefficients are 1 and 1/2. We can get the second from the first by multiplying by 1/2. The next leading coefficient is 1/3; we can get that by multiplying the preceding leading coefficient 1/2 by 2/3. Continuing this pattern, we can get 1/4 by multiplying 1/3 by 3/4. The first coefficient of $P_j(n)$ can be obtained by multiplying the first coefficient of $P_{j-1}(n)$ by $j/(j + 1)$.

Something similar works for the other coefficients, but we have to make an adjustment based on the degree of the term under consideration. For example, consider the coefficient of the third term in each $P_j(n)$, starting with $j = 2$, as listed in

Table 1 Formulas for $P_j(n)$

j	$P_j(n)$ in Factored Form	$P_j(n)$ in Descending Form
0	n	n
1	$n(n + 1)/2$	$(1/2)n^2 + (1/2)n$
2	$n(n + 1)(2n + 1)/6$	$(1/3)n^3 + (1/2)n^2 + (1/6)n$
3	$n^2(n + 1)^2/4$	$(1/4)n^4 + (1/2)n^3 + (1/4)n^2$
4	$n(n + 1)(2n + 1)(3n^2 + 3n - 1)/30$	$(1/5)n^5 + (1/2)n^4 + (1/3)n^3 - (1/30)n$

Mathematizing the World: An Invitation to Modeling

Modeling can mean many things. It is the mathematical representation of real-world behavior, but it also describes relationships among experiment, data, form, and function. What does mathematical modeling mean to you? How does modeling evolve in your classroom? What kinds of tasks invite students into the world of mathematical sense making?

The Editorial Panel of *Mathematics Teacher* invites teachers to share their experiences with modeling. We especially encourage submissions that help *MT* readers wrestle with complexities of mathematical modeling in classrooms. How do we help students discover a dynamic relationship between physical phenomena and mathematical structure?

Here are some prompts to get you started thinking and writing:

- How does your classroom culture promote modeling?
- How does the process of modeling unfold from initial assumptions to final conclusions?
- How do students proceed as they refine their models?
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Table 2 A Selected Term of $P_j(n)$			
j	2	3	4
Third term of $P_j(n)$	$(1/6)n$	$(1/4)n^2$	$(1/3)n^3$

table 2. Following the same approach as before, we can get 1/4 as (1/6)(3/2), and we similarly get 1/3 as (1/4)(4/3). In this case, it seems that the general multiplier should be $j/(j-1)$. Turn next to the coefficients of the second terms. Because each equals 1/2, they can be generated successively by multiplying by 1. But to fit the pattern that is developing, write 1 in the form j/j.

Now consider the three cases together. To get the first coefficient of $P_j(n)$ from the first coefficient of $P_{j-1}(n)$, we multiplied by $j/(j+1)$; to get the second coefficients, we multiplied by j/j; and to get the third coefficients, we multiply by $j/(j-1)$. Once again, a clear pattern emerges. It leads us to predict that the factor for fourth coefficients should be $j/(j-2)$, and that conjecture is indeed correct.

Using these patterns, we can develop a simple algorithm to transform $P_{j-1}(n)$ to $P_j(n)$, as described in Bloom (1993). What we have already discovered allows us to find one coefficient in $P_j(n)$ from each coefficient in $P_{j-1}(n)$. But one additional coefficient in $P_j(n)$ remains to be determined. Using observation 5, this final coefficient can be found by subtracting all the others from 1.

Let's apply this algorithm to derive a formula for $P_5(n)$ using our earlier equation:

$$P_4(n) = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} + (0)n^2 - \frac{n}{30}$$

With $j = 5$, we multiply the leading coefficient by 5/6, the next by 5/5, the next by 5/4, and so on. This process produces coefficients of 1/6, 1/2, 5/12, 0, and –1/12. To get the final coefficient, we compute $1 - 1/6 - 1/2 - 5/12 + 1/12 = 0$. Therefore, we find that

$$P_5(n) = \frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} + (0)n^3 - \frac{n^2}{12} + (0)n.$$

As a test of this result, let's substitute a few values of n. Taking $n = 1$, we find that $P_5(1) = 1$. This result is correct because $P_5(1) = 1^5 = 1$. Next, we try $n = 2$. Our formula for $P_5(n)$ produces $64/6 + 32/2 + 80/12 - 4/12 = 33$, and this result is correct: $P_5(2) = 1^5 + 2^5 = 33$. Readers are encouraged to verify that the formula is correct for $n = 3$ in a similar fashion. Carrying out the algorithm presented above, readers can also find (and test) a formula for $P_6(n)$.

CONCLUSION

A rich literature is associated with sums of integer powers, with many different approaches to computing formulas for Σk^t. Interesting historical connections exist as well, quite a number of which are surveyed by Beery (2009). Of particular note is the work of Jacob Bernoulli (1654–1705) and the development of Bernoulli numbers.

The handshake and cannonball problems that most middle school and high school students know well can lead in a natural way to many interesting extensions. Teachers as well as advanced high school students and college students enjoy the opportunity to investigate these extensions, and the methods presented here provide one avenue for doing so.

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Editor's note: For a different take on the sums of powers, see Somchaipeng, Kruatong, and Panijpan in this issue (pp. 46–50).

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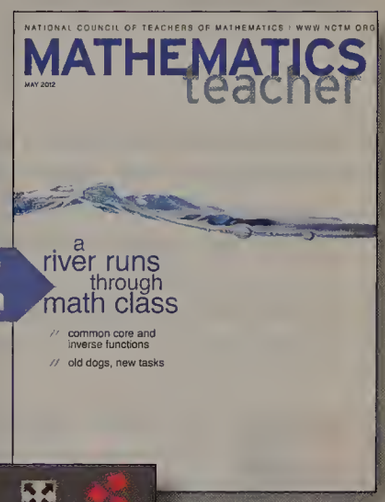
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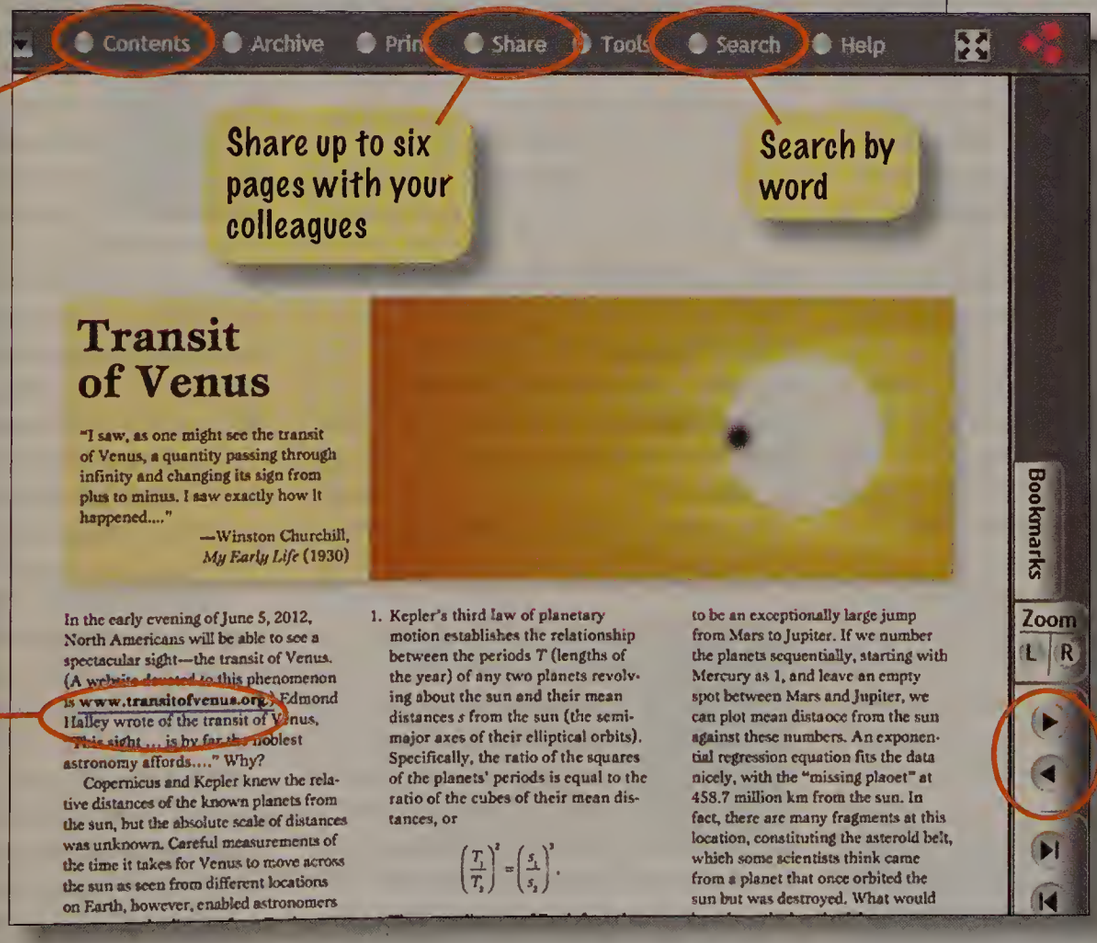
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"I saw, as one might see the transit of Venus, a quantity passing through infinity and changing its sign from plus to minus. I saw exactly how it happened...."
—Winston Churchill, My Early Life (1930)
In the early evening of June 5, 2012, North Americans will be able to see a spectacular sight—the transit of Venus. (A website devoted to this phenomenon is www.transitofvenus.org.) Edmond Halley wrote of the transit of Venus, "this sight ... is by far the noblest astronomy affords...." Why?
Copernicus and Kepler knew the relative distances of the known planets from the sun, but the absolute scale of distances was unknown. Careful measurements of the time it takes for Venus to move across the sun as seen from different locations on Earth, however, enabled astronomers
1. Kepler's third law of planetary motion establishes the relationship between the periods T (lengths of the year) of any two planets revolving about the sun and their mean distances s from the sun (the semi-major axes of their elliptical orbits). Specifically, the ratio of the squares of the planets' periods is equal to the ratio of the cubes of their mean distances, or
$$\left(\frac{T_1}{T_2}\right)^2 = \left(\frac{s_1}{s_2}\right)^3.$$
to be an exceptionally large jump from Mars to Jupiter. If we number the planets sequentially, starting with Mercury as 1, and leave an empty spot between Mars and Jupiter, we can plot mean distaoce from the sun against these numbers. An exponential regression equation fits the data nicely, with the "missing plaoet" at 458.7 million km from the sun. In fact, there are many fragments at this location, constituting the asteroid belt, which some scientists think came from a planet that once orbited the sun but was destroyed. What would

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5 Practices for Orchestrating Productive Mathematics Discussions, Mary Kay Stein and Margaret Smith, 2011. 104 pp., $29.95 paper. ISBN 978-0-87353-677-6. Stock no. 13953. National Council of Teachers of Mathematics and Corwin; www.nctm.org.

When a colleague of mine observes preservice mathematics teachers at work, he frames his reflections with one question: Who is doing the real work here? In classrooms in which there is an understanding that high-quality discussions support student understanding, both students and the teacher must be doing important work. Students must fully engage in cognitively demanding tasks and communicate their mathematical thinking with one another, while teachers are responsible for managing productive discourse based on students' work on these tasks. No small feat, for sure.

Fortunately, this task is made more manageable by Stein and Smith. In *5 Practices for Orchestrating Productive Mathematics Discussions,* they introduce a pedagogical framework to help teachers orchestrate mathematically rich discussions in their classrooms. The five instructional practices presented are informed by research on student thinking as well as years of professional practice.

Preservice and practicing teachers alike should find the text and especially the two highlighted teaching cases, which include classroom transcripts and sample student work, an engaging way to become familiar with the framework. Further, by examining and reflecting on this framework in the context of real practice, teachers can begin to develop specific strategies for shaping classroom discussions that build on their students' thinking.

I strongly recommend this book to any mathematics teacher or teacher educator who is interested in developing the tools to become more effective at orchestrating rich classroom conversations. And I look forward to using this book to inform my own practice.

—Jean McGivney-Burelle
University of Hartford
West Hartford, CT

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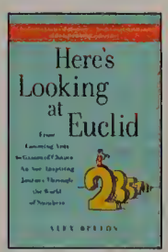
FROM OTHER PUBLISHERS

Here's Looking at Euclid: From Counting Ants to Games of Chance—An Awe-Inspiring Journey through the World of Numbers, Alex Bellows, 2011. 319 pp., $15.00 paper. ISBN 978-1-4165-8828-3. Free Press, Simon and Schuster; www.simonandschuster.com.

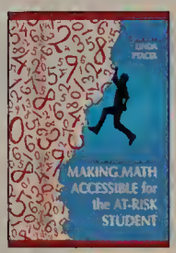


Aptly subtitled, this highly accessible paperback simultaneously informs and entertains on every page. Bellows serves as guide, storyteller, anthropologist, historian, naturalist, and teacher as he explores with amusement and wonder the mathematics behind such diverse topics as ants, baguettes, coral, sudoku, slot machines, and the stock market. I frequently found myself reading passages of this book out loud to my family and highlighting numerous anecdotes and applications to share with my mathematics students.

I enthusiastically recommend this delightful book to anyone with a curious mind.

—Debbie Char
St. Louis Community College
at Forest Park
St. Louis, MO

Making Math Accessible for the At-Risk Student, Grades 7–12, Linda Ptacek, 2011. 184 pp., $40.00 paper; includes CD. ISBN 978-1-59884-590-7. ABC-CLIO; www.abc-clio.com.



Making Math Accessible for the At-Risk Student addresses a varied audience, including middle school and high school prealgebra, algebra, and geometry teachers. This book is a collection of fifty "activities," a term used broadly to include student note sheets, review games, and blackline masters for student worksheets. The CD contains both Microsoft Word and PDF versions of all activities. The Word versions are of great benefit to teachers who want to alter a few problems or directions without having to retype and format an entire page.

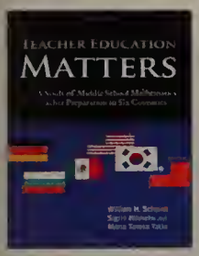
The book begins with a table of activities that is easy to reference for content and format. The introduction, written in a conversational tone, explains how the book was written and who the intended audience is and gives suggestions for its use. Chapters 1–7 are brief overviews (about three pages each) of important insights into issues relating to at-risk students that all teachers should be aware of (such as obstacles to student success) and best practices. Each chapter provides suggestions that refer to

particular activities in the book. Most teaching suggestions that will benefit at-risk students will benefit all students, and Ptacek's experience with special education is obvious.

The chapters are a quick and valuable read, especially for new teachers. Some of Ptacek's activities are similar to traditional worksheets, but the review sheets and games are great. I used the Alpha Math activity as an icebreaker recently, and it really got teachers talking. I recommend this book.

—Christine Latulippe
Norwich University
Northfield, VT

Teacher Education Matters: A Study of Middle School Mathematics Teacher Preparation in Six Countries, William H. Schmidt, Sigrid Blomeke, and Maria Teresa Tatto, 2011. 352 pp., $49.95 paper. ISBN 978-0-8077-5162-6. Teachers College Press; www.tcpress.com.



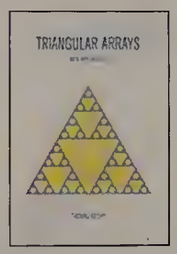
Mathematics Teaching in the 21st Century (MT21), an international study funded by the National Science Foundation, compared the manner in which middle school mathematics teachers are prepared in six selected countries: the United States, Germany, South Korea, Taiwan, Mexico, and Bulgaria. The study, the basis for this book, included a sample of institutions within each country to obtain a representative group to analyze; in all, more than 2600 future mathematics teachers participated. Results pointed toward uniformity among countries with regard to the importance of teaching middle school mathematics but toward wide variety in the manner in which teachers were being prepared for the task.

Educators and policy makers will appreciate the findings of MT21 and its implications for how teacher education programs in this country can be improved. One important suggestion rests on the fact that there is a strong correlation between teacher content knowledge and student achievement, yet U.S. teachers are relatively poorly prepared to teach mathematics at the elementary school and middle school levels.

The study was quite thorough and included detailed analyses of similarities and differences among countries in many areas. Although the book reads like a doctoral dissertation, it does provide valuable insights into how other countries prepare their middle school mathematics teachers in comparison with the United States. Because only six countries were included for analysis, readers cannot make sweeping generalizations from the research. Nonetheless, educators with an interest in teacher preparation will find important information here.

—Jeffrey J. Steckroth
Christopher Newport University
Newport News, VA

Triangular Arrays with Applications, Thomas Koshy, 2011. 656 pp., $98.50 cloth. ISBN 978-0-19-974294-3. Oxford University Press; www.oup.com.



Triangular Arrays with Applications collates and explores a variety of triangular arrays that cover many topics and concepts—combinatorics, probability, number theory, geometry, calculus, and graph theory. Yet even with such diverse coverage, the distinct chapters are unified by the author's weaving of history, applications, examples, and proofs that both challenge and entice readers to learn more. Further, given the complexity of the topics presented here, the book is surprising intelligible and should be understandable to any motivated person who has completed precalculus. However, because of the mathematical register and minimal explanations within the proofs, this book would likely not be fully appreciated without the assistance of a mathematics educator or mathematician.

Nevertheless, this book is an excellent resource for mathematics educators, who can use the triangular arrays and their applications to both motivate and propel students' mathematical knowledge, sense making, proof techniques, and problem-solving skills. I took excerpts from page 8, which focused on binary expansion and the division algorithm, and used these to review simplifying exponents with my class. Students then had fun acting as early computer programmers whose job was to convert randomly selected positive integers into binary digits and check them using the division algorithm.

Beyond helping students, this book could serve as the basis of a master's-level course in a secondary school mathematics education program. It could be especially helpful in preparing future teachers who need a second chance to synthesize old knowledge with nuanced sophistication.

—Paul Madden
Little Wound High School
Kyle, SD

PRODUCTS

PreCalc! and The Tree of Calculus (CDs), Susan E. Cantey, 2009. $10.00 (*PreCalc!* CD) and $12.00 (*The Tree of Calculus* CD). Distributed by CDBaby; www.CDBaby.com.

These CDs—consisting of light-hearted music about learning some of the finer details of high school precalculus and calculus material—are destined to bring smiles to the most veteran mathematics teachers and offer respite from drill-and-kill or chalk-and-talk monotony. Well-priced, the buoyant and delightful songs can be easily imported into one's music player.

Precalc! and *The Tree of Calculus* contain twenty-seven and twenty-six songs, respectively, about precalculus and first-year single-variable calculus. A quick perusal of the playlists catches their flavor: "Transformation Tango," "Stay in the Principle Domain," "The Unit Circle Workout," "Series Are Serious," and "U Substitution." Clearly, the lyricist is a maestro of mathematical wittiness.

For teachers who wish to bring music in from the pedagogical periphery or who believe that mnemonics can help in teaching advanced mathematics, these jingles are destined to please.

—Thomas E. Ricks
Louisiana State University
Baton Rouge

MY FAVORITE lesson

Jan A. Yow

Mathematics Autobiographies

Mathematics anxiety among students and their parents has been documented for years (Fiore 1999; Jackson and Leffingwell 1999). As a classroom teacher, I saw it often in the eyes of my students on the first day of class. Now, as a mathematics teacher educator, I see a passion for mathematics in the eyes of my preservice teachers. How students feel about mathematics makes a difference in how they will learn and how they will teach (Hembree 1990). As a result of research and my own experience, the first assignment that I give students each year is to write a mathematics autobiography. In addition, I ask their parents to write a letter that introduces me to their son or daughter.

The students' autobiographies offer insight into my students as mathematics learners and give me ideas for how to approach future course content. The assignment begins, "Write your mathematics autobiography—your history as a mathematician as a child, as an adolescent, and as a high school student." Specific questions include these:

1. Identify and discuss significant moments in your mathematical life, from kindergarten to the present. Were these experiences positive or negative? Why?
2. Describe any factors, teachers, or other people who have influenced your desire or lack of desire to learn mathematics.
3. How do you learn best? How do you learn mathematics best?

The assignment typically includes six to nine questions, all of which students must answer. The length of responses varies from one to three pages. I ask students to submit two copies (one with their name and one without their name); I tell them that we may share the anonymous copies with one another in class. Students can request that their autobiography not be shared if they want to include information that they want only me to read. This exchange invites students to see how their experiences in mathematics compare with those of their peers.

A companion to the students' autobiography is the letter I ask parents or caregivers to write about their child. This piece is optional; the student is not punished if the parent is unable to complete the assignment. I ask parents to include anything in the letter that they think would help me teach mathematics to their child more effectively. Most often, parents tell me about sports or extracurricular activities that interest their child. They share with me grade levels or previous courses that have affected their child's mathematical history, both positively and negatively, and why. I have even had parents tell me that their child has a history of *not* doing his or her homework and include a request for me to keep them informed if this pattern continues.

The insider perspective on my students that the autobiography and the letter provide is invaluable when we begin our relationship as learners and teachers of mathematics. This assignment creates an opportunity for students to write in a mathematics classroom and introduces me to the mathematicians in my room through their own words and through the words of their parents.

I have used this lesson with students in a wide range of classes, from algebra to precalculus classes to graduate mathematics education courses. The return rate has always been high; the positive effect associated with the assignment proves a credible means to evaluate and address the mathematics anxiety level in my classes at the start of the year. And throughout the year, as I plan lessons and work with parents, I find myself referring back to these autobiographies and letters.

ACKNOWLEDGMENTS

Special thanks are due Carol E. Malloy, whose autobiographical activity in her university methods course was a basis for these autobiographical questions.

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The Back Page provides a forum for readers to share a favorite lesson. Lessons to be considered for publication should be submitted to mt.msubmit.net. Lessons should not exceed 600 words and are subject to abridgment.

Edited by ***Jennifer Wexler,*** *wexlerj@newtrier.k12.il.us, New Trier High School, Winnetka, IL*

JAN A. YOW, jyow@sc.edu, is an assistant professor of mathematics education at the University of South Carolina in Columbia. A National Board Certified Teacher, she is interested in mathematics teacher leadership and the teaching and learning of mathematics for all students.

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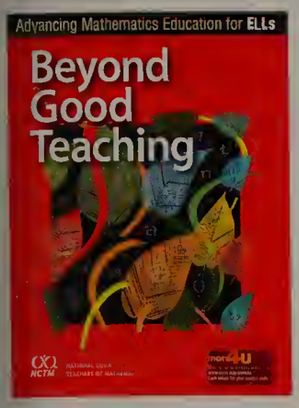

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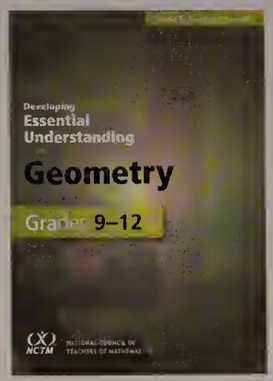

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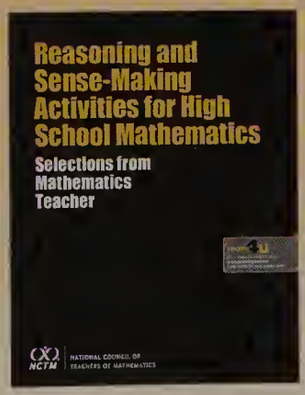

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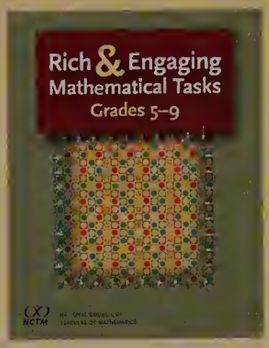

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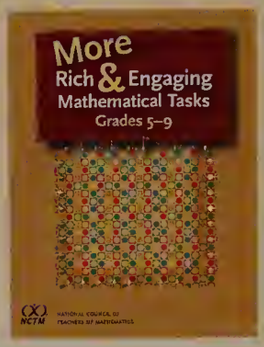

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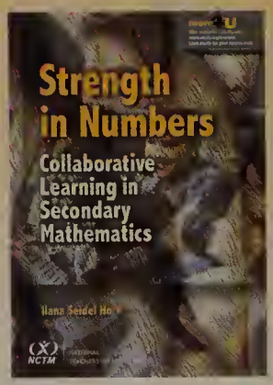

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